Kähler Ricci flow on Fano manfiolds(I)

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Abstract

We study the evolution of anticanonical line bundles along the Kähler Ricci flow. We show that under some conditions, the convergence of Kähler Ricci flow is determined by the properties of the anticanonical divisors of M. As examples, the Kähler Ricci flow on M converges when M is a Fano surface and $c_1^2(M) = 1$ or $c_1^2(M) = 3$. Combined with the work in [CW1] and [CW2], this gives a Ricci flow proof of the Calabi conjecture on Fano surfaces with reductive automorphism groups. The original proof of this conjecture is due to Gang Tian in [Tian90].

Contents

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1 Introduction

In this paper, we introduce a new criteria for the convergence of the Kähler Ricci flow in general Fano manifolds. This might be useful in attracting renewed attentions to the renown Calabi conjecture in higher dimensional Fano manifolds. Moreover, we verify these criteria for the Kähler Ricci flow in Fano surfaces $\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$. Consequently, we give a proof of the convergence of the Kähler Ricci flow on such Fano surfaces. The existence of KE (Kähler Einstein) metrics on these Fano surfaces follows as a corollary.

In Kähler geometry, a dominating problem is to prove the celebrated Calabi conjecture ([Ca]). It states that if the first Chern class of a Kähler manifold M is positive, null or negative, then the canonical Kähler class of M admits a KE metric. In 1976, the null case Calabi conjecture was proved by S. T. Yau. Around the same time, the negative case was proved independently by T. Aubin and S. T. Yau. However, the positive first Chern class case is much more complicated. In [Ma], Matsushima showed that the reductivity of $Aut(M)$ is a necessary condition for the existence of KE metric. In [Fu], A. Futaki introduced an algebraic invariant which vanishes if the canonical Kähler class admits a KE metric. Around 1988, G. Tian [Tian90] proved the Calabi conjecture for Fano surfaces with reductive automorphism groups. By the classification theory of complex surfaces, Tian actually proved the existence of KE metric on Fano surface M whenever M is diffeomorphic to $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ $(3 \leq k \leq 8)$. Prior to Tian's work, there is a series of important works in [Tian87], [TY], [Siu] where existence results of KE metrics on some special complex surfaces were derived.

Let $(M, [\omega])$ be a Fano manifold where $[\omega]$ is the canonical Kähler class. Suppose that $\{\omega_t\}(t \in [0, \infty))$ is the one parameter family of Kähler metrics in $[\omega]$ evolves under the Kähler Ricci flow. Let K_M^{-1} be the anticanonical line bundle equipped with a natural, evolving metric $h_t = \omega_t^n = \det g_{\omega_t}$. In this paper, we adopt the view that one must study the the associated evolution of line bundles when study Kähler Ricci flow :

$$\begin{pmatrix} (K_M^{-\nu}, h_t^{\nu}) \\ \downarrow \\ (M, \omega_t) \end{pmatrix}.$$

It is well known that $K_M^{-\nu}$ is very ample when ν is large. Let $N_\nu = \dim H^0(K_M^{-\nu}) - 1$, $\{S_{\nu,\beta}^t\}_{\beta=0}^{N_\nu}$ be orthonormal holomorphic sections in $H^0(K_M^{-\nu})$ with respect to metric g_t and h_t^ν, i.e.,

$$\int_M \left\langle S_{\nu,\alpha}^t, S_{\nu,\beta}^t \right\rangle_{h_t^\nu} \omega_t^n = \delta_{\alpha\beta}.$$

An easy observation shows that

$$F_\nu(x,t) = \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} \left| S_{\nu,\beta}^t \right|_{h_t^\nu}^2 (x)$$

is a well defined function on $M \times [0, \infty)$ (independent of the choice of orthonormal basis). The Kähler Ricci flow $\{(M, g(t)), 0 \leq t < \infty\}$ is called a **flow tamed by** ν if $K_M^{-\nu}$ is very ample and $F_\nu(x,t)$ is a uniformly bounded function on $M \times [0, \infty)$. The flow is called a **tamed Kähler Ricci flow** if it is tamed by a some integer ν.

For a tamed flow, we can reduce the convergence of the flow to the values of local α-invariants of plurianticanonical divisors.

Definitionin 1. *Suppose L is a line bundle over M with Hermitian metric h, S is a holomorphic section of L, $x \in M$. Define*

$$\alpha_x(S) = \sup\{\alpha | \|S\|_h^{-2\alpha} \textit{ is locally integrable around } x\}.$$

See [Tian90] and [Tian91] for more details about this definition. Note that $\alpha_x(S)$ is also called singularity exponent ([DK]), logarithm canonical threshold ([ChS]), etc. It is determined only by the singularity type of $Z(S)$. Therefore, if $S \in H^0(K_M^{-\nu})$, $\alpha_x(S)$ can only achieve finite possible values.

Definitionin 2. *Let $\mathscr{P}_{G,\nu,k}(M,\omega)$ be the collection of all G-invariant functions of form $\frac{1}{\nu}\log(\sum_{\beta=0}^{k-1}\|\tilde{S}_{\nu,\beta}\|_{h^\nu}^2)$, where $\{\tilde{S}_{\nu,\beta}\}_{\beta=0}^{k-1}$ $(1 \leq k \leq \dim H^0(K_M^{-\nu}))$ satisfies*

$$\int_M \langle \tilde{S}_{\nu,\alpha}, \tilde{S}_{\nu,\beta}\rangle_{h^\nu}\omega^n = \delta_{\alpha\beta}, \quad 0 \leq \alpha, \beta \leq k-1 \leq \dim(K_M^{-\nu}) - 1; \quad h = \det g_\omega.$$

Define

$$\alpha_{G,\nu,k} \triangleq \sup\{\alpha | \sup_{\varphi \in \mathscr{P}_{G,\mu,k}} \int_M e^{-\alpha\varphi}\omega^n < \infty\}.$$

If G is trivial, we denote $\alpha_{\nu,k}$ as $\alpha_{G,\nu,k}$.

It turns out that the value of local α-invariants, $\alpha_{\nu,1}$ and $\alpha_{\nu,2}$ play important roles in the convergence of Kähler Ricci flow.

Theorem 1. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν. If $\alpha_{\nu,1} > \frac{n}{(n+1)}$, then φ is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.*

Theorem 2. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν. If $\alpha_{\nu,2} > \frac{n}{n+1}$ and $\alpha_{\nu,1} > \frac{1}{2 - \frac{n-1}{(n+1)\alpha_{\nu,2}}}$, then φ is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.*

In fact, if a Kähler Ricci flow is tamed by some large ν, an easy argument (c.f. Section 2.3) shows that the following strong partial C^0-estimate hold.

$$\left|\varphi(t) - \sup_M \varphi(t) - \frac{1}{\nu}\log\sum_{\beta=0}^{N_\nu}\left|\lambda_\beta(t)\tilde{S}_{\nu,\beta}^t\right|_{h_0^\nu}^2\right| < C. \tag{1}$$

Here $\varphi(t)$ is the evolving Kähler potential. $0 < \lambda_0(t) \leq \lambda_1(t) \leq \cdots \leq \lambda_{N_\nu}(t) = 1$ are $N_\nu + 1$ positive functions of time t. $\{\tilde{S}^t_{\nu,\beta}\}_{\beta=0}^{N_\nu}$ is an orthonormal basis of $H^0(K_M^{-\nu})$ under the fixed metric g_0. Intuitively, inequality (1) means that we can control $Osc_M \varphi(t)$ by $\frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} \left| \lambda_\beta(t) \tilde{S}^t_{\nu,\beta} \right|^2_{h_0^\nu}$ which only blows up along intersections of pluri-anticanonical divisors. Therefore, the estimate of $\varphi(t)$ is more or less translated to the study of the property of pluri-anticanonical holomorphic sections.

In view of these theorems, we need to check the following two conditions:

- Whether the Kähler Ricci flow is a tamed flow;
- Whether the $\alpha_{\nu,k}$ $(k = 1, 2)$ are big enough.

The second condition can be checked by purely algebraic geometry method. The first condition is much weaker. We believe that it holds for every Kähler Ricci flow on Fano manifold although we cannot prove this right now. However, under some extra conditions, we can check the first condition by the following theorem.

Theorem 3. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow satisfying the following conditions.*

- *volume ratio bounded from above, i.e., there exists a constant K such that*

$$\mathrm{Vol}_{g(t)}(B_{g(t)}(x, r)) \leq K r^{2n}$$

  for every geodesic ball $B_{g(t)}(x, r)$ satisfying $r \leq 1$.

- *weak compactness, i.e., for every sequence $t_i \to \infty$, by passing to subsequence, we have*

$$(M, g(t_i)) \xrightarrow{C^\infty} (\hat{M}, \hat{g}),$$

  where $(\hat{M}, \hat{g})$ is a Q-Fano normal variety, $\xrightarrow{C^\infty}$ means Cheeger-Gromov convergence, i.e., $(M, g(t_i))$ converges to $(\hat{M}, \hat{g})$ in Gromov-Hausdorff topology, and the convergence is in smooth topology away from singularities.

Then this flow is tamed.

In the case of Fano surfaces under continuous path, Tian proved a similar theorem. However, in his proof, every metric is a Kähler Einstein metric, so there are more estimates available. In particular, Ricci curvature is uniformly bounded there. Our proof here is more technical since we have no Ricci curvature control. The concept of Q-Fano variety is first defined in [DT], it's a natural generalization of Fano manifold. A Q-Fano variety is an algebraic variety with a very ample line bundle whose restriction on the smooth part is the plurianticanonical line bundle. In the proof of this theorem, Hörmander's

L^2-estimate of $\bar{\partial}$-operator, Perelman's fundamental estimates and the uniform control of Sobolev constants (c.f. [Ye], [Zhq]) play crucial roles. Actually, Sobolev constants control and Perelman's estimates assure the uniform control of $\||S|_{h_t^\nu}\|_{C^0(M)}$ and $\||\nabla S|_{h_t^\nu}\|_{C^0(M)}$ whenever S is a unit norm holomorphic section in $H^0(K_M^{-\nu})$. Hörmander's L^2-estimate of $\bar{\partial}$-operator assures that the plurigenera is continuous under the sequential convergence. Therefore, for every fixed ν, we have

$$\lim_{i\to\infty}\inf_{x\in M} e^{\nu F_\nu(x,t_i)} = \lim_{i\to\infty}\inf_{x\in M}\sum_{\beta=0}^{N_\nu}\left|S_{\nu,\beta}^{t_i}\right|^2_{h_{t_i}^\nu}(x) = \inf_{x\in\hat{M}}\sum_{\beta=0}^{N_\nu}\left|\hat{S}_{\nu,\beta}\right|^2_{\hat{h}^\nu}(x).$$

This equation relates the tamed condition to the property of every limit space. If every limit space is a Q-Fano normal variety, we know

$$\inf_{x\in\hat{M}}\sum_{\beta=0}^{N_\nu}\left|\hat{S}_{\nu,\beta}\right|^2_{h_{t_i}^\nu}(x) > 0$$

for some ν depending on $\hat{M}$. Then a contradiction argument can show that $e^{\nu F_\nu}$ must be uniformly bounded from below for some large ν. In other words, F_ν is uniformly bounded (the upper bound of F_ν is a corollary of the boundedness of $\||S|_{h_t^\nu}\|_{C^0(M)}$) and the flow is tamed.

As applications of Theorem 1 to Theorem 3, we can show the convergence of Kähler Ricci flow on Fano surface M when $c_1^2(M) \leq 4$. Actually, in [CW3], we proved the weak compactness of 2-dimensional Kähler Ricci flow.

Lemma 1. *([CW3]) Suppose $\{(M, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution on a Fano surface. Then for any sequence $t_i \to \infty$, we have Cheeger-Gromov convergence*

$$(M, g(t_i)) \stackrel{C^\infty}{\to} (\hat{M}, \hat{g})$$

where $(\hat{M}, \hat{g})$ is a Kähler Ricci soliton orbifold with finite singularities. In particular, $\hat{M}$ is Q-Fano normal variety.

Moreover, the volume ratio upper bound is proved in the process of proving weak compactness. Therefore, Theorem 3 and Lemma 1 implies that every 2-dimensional Kähler Ricci flow is a tamed flow. The authors remark that, in an unpublished work (c.f. [Se1], [FZ]), Tian has pointed out earlier the sequential convergence of the 2-dimensional Kähler Ricci flow to Kähler Ricci soliton orbifolds under the Gromov-Hausdorff topology. Under the extra condition that Ricci curvature is uniformly bounded along the flow, Lemma 1 was proved by Natasa Sesum in [Se1]. However, for our purpose of using Theorem 3, these convergence theorems are not sufficient (We need Cheeger-Gromov convergence without Ricci curvature bound condition). In the course of proof of this lemma, the fundamental work of G. Perelman on the Ricci flow (non-local collapsing theorem, pseudo-locality theorem and canonical neighborhood theorem)play critical roles. Under some geometric constraints natural to our setting, we proved an inverse pseudo-locality

theorem (heuristically speaking, no "bubble" will disappear suddenly). For that purpose, we need to have a uniform control of volume growth on all scales. We found that the argument for volume ratio upper bound in the beautiful work [TV1] and [TV2] is very enlightening.

In order to show the convergence of a 2-dimensional Kähler Ricci flow, we now only need to see if $\alpha_{\nu,k}(k=1,2)$ are big enough to satisfy the requirements of Theorem 1 or Theorem 2. If $c_1^2(M) \leq 4$, one can show that either Theorem 1 or Theorem 2 applies. However, the convergence of Kähler Ricci flow on Fano surfaces M are proved in [CW2] when $c_1^2(M) = 2$ or $c_1^2(M) = 4$. The only remained cases are $c_1^2(M) = 1$ and $c_1^2(M) = 3$. So we concentrate on these two cases and have the following lemma.

Lemma 2. *Suppose M is a Fano surface, ν is any positive integer.*

- *If $c_1^2(M) = 1$, then $\alpha_{\nu,1} \geq \frac{5}{6}$.*
- *If $c_1^2(M) = 3$, then $\alpha_{\nu,1} \geq \frac{2}{3}$, $\alpha_{\nu,2} > \frac{2}{3}$.*

Actually, the value of $\alpha_{\nu,1}$ was calculated by Ivan Cheltsov (c.f. [Chl]) for every Fano surface. The value of $\alpha_{\nu,2}$ was also calculated for every cubic surface ($c_1^2(M) = 3$) by Yalong Shi (c.f. [SYl]). For the convenience of readers, we give an elementary proof at the end of this paper.

Therefore, Theorem 1 and Theorem 2 applies respectively and show the existence of KE metrics on M whenever $c_1^2(M) = 1$ or 3. Combining this result with the results we proved in [CW1] and [CW2], we can give an alternative proof of the celebrated theorem of Tian:

Theorem ([Tian90])**.** *A Fano surface M admits a Kähler Einstein metric if and only if $Aut(M)$ is reductive.*

This solved a famous problem of Calabi for Fano surfaces [Ca]. This work of Tian clearly involves deep understanding of many aspects of Kähler geometry as well as its intimate connection to algebraic geometry. It is one of the few highlights in Kähler geometry which deserve new proofs by Ricci flow. On the other hand, the Kähler Ricci flow is a natural way to understand Calabi conjecture in Fano setting. Following Yau's estimate ([Yau78]), H. D. Cao ([Cao85]) proved that the Kähler Ricci Flow with smooth initial metric always exists globally. On a KE manifold, the first named author and Tian showed that Kähler Ricci Flow converges exponentially fast toward the KE metric if the initial metric has positive bisectional curvature (c.f. [CT1], [CT2]). Using his famous μ-functional, Perelman proved that scalar curvature, diameter and normalized Ricci potentials are all uniformly bounded along Kähler Ricci flow (c.f. [SeT]). These fundamental estimates of G. Perelman opens the door for a more qualitative analysis of singularities formed in the Kähler Ricci flow. As a corollary of his estimates, G. Perelman announced that the Kähler Ricci flow will always converge to the KE metric on every KE manifold. The first written

proof of this statement appeared in [TZ] where Tian and Zhu also generalized it to Kähler manifolds admitting Kähler Ricci solitons. In our humble view, the estimates of G. Perelman makes the flow approach a plausible one in terms of understanding Calabi conjecture in Fano setting. We hope that this modest progress in Kähler Ricci flow will attract more attentions to the renown Hamilton-Tian conjecture. Namely, any Kähler Ricci flow will converge to some Kähler Ricci solitons with mild singularities in Cheeger-Gromov topology, perhaps of different complex structures.

The application of strong partial C^0-estimate is one of the crucial components of this paper. It sets up the frame work of our proof for the convergence of 2-dimensional Kähler Ricci flow. This estimate originates from the strong partial C^0-estimate along continuous path in Tian's original proof (c.f. [Tian90]). He also conjectured that the strong partial C^0-estimate holds along continuous path in higher dimensional Kähler manifolds (c.f. [Tian91]).

The organization of this paper is as follows. In section 2, along each tamed flow, we reduce the C^0-estimate of the potential function φ to the calculation of local α-invariants of sections $S \in H^0(K_M^{-\nu})$. In section 3, we study the basic properties of pluri-anticanonical holomorphic sections along Kähler Ricci flow. Here we discuss the applications of Hörmander's L^2-estimate of $\bar{\partial}$-operator and we deduce the uniform bounds of $|S|_{h_t^\nu}$ and $|\nabla S|_{h_t^\nu}$. Using these estimates, we give a justification theorem of the tamed condition. In section 4, we calculate $\alpha_{\nu,k}(M)(k=1,2)$ when $c_1^2(M)=1$ or 3 and show that their values are big enough to obtain the C^0-estimate of the evolving potential function $\varphi(t)$.

Remark. *In the subsequent paper [Wang], we will apply these methods to Kähler Ricci flow on orbifold Fano surfaces. As an application, we find some new Kähler Einstein orbifolds. In particular, we prove the following conjecture (c.f. [Kosta]). Let Y be a degree 1 del Pezzo surface having only Du Val singularities of type $\mathbb{A}_n$ for $n \leq 6$, then Y admits a Kähler Einstein metric.*

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2 Estimates Along Kähler Ricci Flow

2.1 Basic Kähler Geometry

Let M be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on M. In local coordinates $z_1, \cdots, z_n$, this ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} d\, z^i \wedge d\, z^{\bar{j}} > 0,$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive (1,1)-form. Given a Kähler metric ω, its volume form is

$$\omega^n = \frac{1}{n!} \left(\sqrt{-1}\right)^n \det\left(g_{i\bar{j}}\right) d\, z^1 \wedge d\, z^{\bar{1}} \wedge \cdots \wedge d\, z^n \wedge d\, z^{\bar{n}}.$$

The curvature tensor is

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}, \qquad \forall\, i, j, k, l = 1, 2, \cdots n.$$

The Ricci curvature form is

$$\mathrm{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}}(\omega) d\, z^i \wedge d\, z^{\bar{j}} = -\sqrt{-1}\partial\bar{\partial} \log\ \det(g_{k\bar{l}}).$$

It is a real, closed (1,1)-form and $[Ric] = 2\pi c_1(M)$.

From now on we we assume M has positive first Chern class, i.e., $c_1(M) > 0$. We call $[\omega]$ as a canonical Kähler class if $[\omega] = [Ric] = 2\pi c_1(M)$. If we require the initial metric is in canonical class, then the normalized Ricci flow (c.f. [Cao85]) on M is

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \qquad \forall\, i,\ j = 1, 2, \cdots, n. \tag{2}$$

Denote $\omega = \omega_{g(0)}$, $\omega_{g(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$. φ_t is called the Kähler potential and sometime it is denoted as φ for simplicity. On the level of Kähler potentials, Kähler Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{{\omega_\varphi}^n}{\omega^n} + \varphi + u_\omega, \tag{3}$$

where u_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = -\sqrt{-1}\partial\bar{\partial} u_\omega, \text{ and } \int_M (e^{-u_\omega} - 1)\omega^n = 0.$$

As usual, the flow equation (2) or (3) is referred as the Kähler Ricci flow in canonical class of M. It is proved by Cao [Cao85], who followed Yau's celebrated work [Yau78], that this flow exists globally for any smooth initial Kähler metric in the canonical class.

$$
\boxed{\begin{aligned}
\frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}} &= \triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}}R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}}R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}}R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} \\
&\quad -\tfrac{1}{2}\left(R_{i\bar{p}}R_{p\bar{j}k\bar{l}} + R_{p\bar{j}}R_{i\bar{p}k\bar{l}} + R_{k\bar{p}}R_{i\bar{j}p\bar{l}} + R_{p\bar{l}}R_{i\bar{j}k\bar{p}}\right). \\
\frac{\partial}{\partial t}R_{i\bar{j}} &= \triangle R_{i\bar{j}} + R_{i\bar{j}p\bar{q}}R_{q\bar{p}} - R_{i\bar{p}}R_{p\bar{j}}. \\
\frac{\partial}{\partial t}R &= \triangle R + R_{i\bar{j}}R_{j\bar{i}} - R.
\end{aligned}}
$$

Table 1: Curvature evolution equations along Kähler Ricci flow

In this note, we only study Kähler Ricci flow in the canonical class. For the simplicity of notation, we may not mention that the flow is in canonical class every time.

Along Kähler Ricci flow , the evolution equations of curvatures are listed in Table 1.

Let $\mathscr{P}(M,\omega) = \{\varphi | \omega + \sqrt{-1}\partial\bar{\partial}\varphi\}$. It is shown in [Tian87] that there is a small constant $\delta > 0$ such that

$$\sup_{\varphi \in \mathscr{P}(M,\omega)} \frac{1}{V}\int_M e^{-\delta(\varphi - \sup_M \varphi)}\omega^n < \infty.$$

The supreme of such δ is called the α-invariant of (M,ω) and it is denoted as $\alpha(M,\omega)$. Let G be a compact subgroup of $Aut(M)$ and ω is a G-invariant form. We denote

$$\mathscr{P}_G(M,\omega) = \{\varphi | \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \varphi \text{ is invariant under } G\}.$$

Similarly, we can define $\alpha_G(M,\omega)$. Actually, $\alpha_G(M,\omega)$ is an algebraic invariant. It is called global log canonical threshold $lct(X,G)$ by algebraic geometers. See [ChS] for more details.

2.2 Known Estimates along General Kähler Ricci Flow

There are a lot of estimates along Kähler Ricci flow in the literature. We list some of them which are important to our arguments.

Proposition 2.1 (Perelman, c.f. [SeT]). *Suppose $\{(M^n, g(t)), 0 \le t < \infty\}$ is a Kähler Ricci flow solution. There are two positive constants $\mathcal{B}, \kappa$ depending only on this flow such that the following two estimates hold.*

1. *Under metric $g(t)$, let R be the scalar curvature, $-u$ be the normalized Ricci potential, i.e.,*

$$Ric - \omega_{\varphi(t)} = -\sqrt{-1}\partial\bar{\partial}u, \quad \frac{1}{V}\int_M e^{-u}\omega^n_{\varphi(t)} = 1.$$

Then we have

$$\|R\|_{C^0} + \operatorname{diam} M + \|u\|_{C^0} + \|\nabla u\|_{C^0} < \mathcal{B}.$$

2. Under metric $g(t)$, $\dfrac{\mathrm{Vol}(B(x,r))}{r^{2n}} > \kappa$ for every $r \in (0,1)$, $(x,t) \in M \times [0,\infty)$.

After this fundmental work of G. Perelman, many interesting papers appear during this period. We include a few references here for the convenience of readers: [CH], [CST], [CW1], [CW2], [Hei], [PSS], [PSSW1], [PSSW2], [Ru], [RZZ], [Se1], [Se2], [TZs],etc. In this subsection, we cite a few results below which are directly related to our work here.

Proposition 2.2 ([Zhq], [Ye])**.** *There is a uniform Sobolev constant C_S along the Kähler Ricci flow solution $\{(M^n, g(t)), 0 \le t < \infty\}$. In other words, for every $f \in C^\infty(M)$, we have*

$$(\int_M |f|^{\frac{2n}{n-1}} \omega_\varphi^n)^{\frac{n-1}{n}} \le C_S \{\int_M |\nabla f|^2 \omega_\varphi^n + \frac{1}{V^{\frac{1}{n}}} \int_M |f|^2 \omega_\varphi^n\}.$$

Proposition 2.3 (c.f. [TZ])**.** *There is a uniform weak Poincarè constant C_P along the Kähler Ricci flow solution $\{(M^n, g(t)), 0 \le t < \infty\}$. Namely, for every nonnegative function $f \in C^\infty(M)$, we have*

$$\frac{1}{V}\int_M f^2 \omega_\varphi^n \le C_P \{\frac{1}{V}\int_M |\nabla f|^2 \omega_\varphi^n + (\frac{1}{V}\int_M f \omega_\varphi^n)^2\}.$$

As an easy application of a normalization technique initiated in [CT1], one can prove the following property.

Proposition 2.4 (c.f. [PSS], [CW2])**.** *By properly choosing initial condition, we have*

$$\|\dot{\varphi}\|_{C^0} + \|\nabla \dot{\varphi}\|_{C^0} < C$$

for some constant C independent of time t.

Based on these estimates, the authors proved the following properties.

Proposition 2.5 (c.f. [Ru], [CW2])**.** *There is a constant C such that*

$$\frac{1}{V}\int_M (-\varphi)\omega_\varphi^n \le n \sup_M \varphi - \sum_{i=0}^{n-1} \frac{i}{V}\int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i} + C. \tag{4}$$

In particular, we have

$$\frac{1}{V}\int_M (-\varphi)\omega_\varphi^n \le n \sup_M \varphi + C. \tag{5}$$

Proposition 2.6 (c.f. [Ru], [CW2])**.** *For every δ less than the α-invariant of M, there is a uniform constant C such that*

$$\sup_M \varphi < \frac{1-\delta}{\delta} \int_M (-\varphi)\omega_\varphi^n + C \tag{6}$$

along the flow.

Lemma 2.1 (c.f. [Ru], [CW2])**.** *Along Kähler Ricci flow $\{(M^n, g(t)), 0 \le t < \infty\}$ in the canonical class of Fano manifold M, the following conditions are equivalent.*

- *φ is uniformly bounded.*
- *$\sup_M \varphi$ is uniformly bounded from above.*
- *$\inf_M \varphi$ is uniformly bounded from below.*
- *$\int_M \varphi\omega^n$ is uniformly bounded from above.*
- *$\int_M (-\varphi)\omega_\varphi^n$ is uniformly bounded from above.*
- *$I_\omega(\varphi)$ is uniformly bounded.*
- *$Osc_M \varphi$ is uniformly bounded.*

As a simple corollary, we have

Theorem 2.1 ([CW2])**.** *If $\alpha_G(M,\omega) > \frac{n}{n+1}$ for some G-invariant metric ω, then φ is uniformly bounded along the Kähler Ricci flow initiating from ω.*

2.3 Estimates along Tamed Kähler Ricci Flow

In this section, we only study tamed flow.

Definition 2.1. *For every positive integer ν, we can define a function F_ν on spacetime $M \times [0,\infty)$ as follows.*

$$F_\nu(x,t) \triangleq \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} \left|S_{\nu,\beta}^t\right|_{h_t^\nu}^2 (x)$$

where $\{S_{\nu,\beta}^t\}_{\beta=0}^{N_\nu}$ is an orthonormal basis of $H^0(K_M^{-\nu})$ under the metric $g_t = g(t)$ and $h_t^\nu = (\det g_t)^\nu$, i.e.,

$$\int_M \langle S_{\nu,\alpha}^t, S_{\nu,\beta}^t \rangle_{h_t^\nu} \omega_t^n = \delta_{\alpha\beta}, \quad N_\nu = \dim H^0(K_M^{-\nu}) - 1.$$

Note that this definition is independent of the choice of orthonormal basis of $H^0(K_M^{-\nu})$.

Definition 2.2. *$\{(M^n, g(t)), 0 \le t < \infty\}$ is called a tamed flow if there is a big integer ν such that the following properties hold.*

- *$K_M^{-\nu}$ is very ample.*
- *$|F_\nu|_{C^0(M\times[0,\infty))} < \infty$.*

Suppose $\{(M^n, g_t), 0 \le t < \infty\}$ is a tamed flow. Under the metric g_t and h_t^ν, we choose $\{S_{\nu,\beta}^t\}_{\beta=0}^N$ as an othonormal basis of $H^0(K_M^{-\nu})$. At the same time, let $\{\tilde{S}_{\nu,\beta}^t\}_{\beta=0}^N$ be an orthonormal basis of $H^0(K_M^{-\nu})$ under the metric g_0 and h_0^ν. Then we have two embeddings.

$$\begin{aligned}
\Phi^t : M \mapsto \mathbb{CP}^N, \quad & x \mapsto [S_{\nu,0}^t(x) : \cdots : S_{\nu,N}^t(x)];\\
\Psi^t : M \mapsto \mathbb{CP}^N, \quad & x \mapsto [\tilde{S}_{\nu,0}^t(x) : \cdots : \tilde{S}_{\nu,N}^t(x)].
\end{aligned}$$

By rotating basis if necessary, we can assume $\Phi^t = \sigma(t) \circ \Psi^t$ where

$$\sigma(t) = a(t) diag\{\lambda_0(t), \cdots, \lambda_N(t)\}, \quad 0 < a(t), \quad 0 < \lambda_0(t) < \lambda_1(t) < \cdots < \lambda_N(t) = 1.$$

This indicates that Kähler Ricci flow equation can be rewritten as

$$\begin{aligned}
\dot{\varphi} &= \log \frac{\omega_\varphi^n}{\omega^n} + \varphi + u_\omega\\
&= \frac{1}{\nu} \log \frac{\sum_{\beta=0}^N \left|S_{\nu,\beta}^t\right|_{h_t^\nu}^2}{\sum_{\beta=0}^N \left|S_{\nu,\beta}^t\right|_{h_0^\nu}^2} + \varphi + u_\omega\\
&= \frac{1}{\nu} \log \sum_{\beta=0}^N \left|S_{\nu,\beta}^t\right|_{h_t^\nu}^2 - \frac{1}{\nu} \log \sum_{\beta=0}^N \left|a(t)\lambda_\beta(t)\tilde{S}_{\nu,\beta}^t\right|_{h_0^\nu}^2 + \varphi + u_\omega\\
&= F_\nu(x,t) - \frac{1}{\nu} \log \sum_{\beta=0}^N \left|\lambda_\beta(t)\tilde{S}_{\nu,\beta}^t\right|_{h_0^\nu}^2 + \varphi + u_\omega - \frac{2}{\nu}\log a(t).
\end{aligned}$$

In other words,

$$\varphi - \frac{2}{\nu}\log a(t) = \dot{\varphi} - u_\omega - F_\nu(x,t) + \frac{1}{\nu} \log \sum_{\beta=0}^N \left|\lambda_\beta(t)\tilde{S}_{\nu,\beta}^t\right|_{h_0^\nu}^2.$$

Since $F_\nu(x,t)$, $\dot{\varphi}$ and u_ω are all uniformly bounded, we obtain

$$\varphi - \frac{2}{\nu}\log a(t) \sim \frac{1}{\nu} \log \sum_{\beta=0}^N \left|\lambda_\beta(t)\tilde{S}_{\nu,\beta}^t\right|_{h_0^\nu}^2.$$

Here we use the notation $\sim$ to denote that the difference of two sides are controlled by a constant. It follows that

$$\varphi - \sup_M \varphi \sim \frac{1}{\nu} \log \sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu} - \frac{1}{\nu} \sup_M \log \sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu}.$$

It is obvious that

$$\sup_M \log \sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu} \leq \sup_M \log \sum_{\beta=0}^{N} \left|\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu} < C.$$

On the other hand, we have

$$\begin{aligned}
\sup_M \log \sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu} &\geq \sup_M \log \left|\lambda_N(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu} \\
&= \sup_M \log \left|\tilde{S}^t_{\nu,N}\right|^2_{h_0^\nu} \\
&= \log \sup_M \left|\tilde{S}^t_{\nu,N}\right|^2_{h_0^\nu} \\
&\geq \log \frac{1}{V} \int_M \left|\tilde{S}^t_{\nu,N}\right|^2_{h_0^\nu} \omega^n \\
&= -\log V.
\end{aligned}$$

Therefore, $\frac{1}{\nu} \sup_M \log \sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu}$ is uniformly bounded and it yields that

$$\varphi - \sup_M \varphi \sim \frac{1}{\nu} \log \sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu}. \tag{7}$$

So we have proved the following property.

Proposition 2.7. *If $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν, then there is a constant C (depending on this flow and ν) such that*

$$|\varphi - \sup_M \varphi - \frac{1}{\nu} \log \sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu}| < C \tag{8}$$

uniformly along this flow.

Inequality (8) is called the strong partial C^0-estimate by Tian. Using this estimate, the control of $\|\varphi\|_{C^0(M)}$ along Kähler Ricci flow is reduced to the control of values of local α-invariants (See Definition 1 and Definition 2) of holomorphic sections $S \in H^0(K_M^{-\nu})$.

Theorem 2.2. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν. If $\alpha_{\nu,1} > \frac{n}{n+1}$, then φ is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.*

Proof. Suppose not. Then there is a sequence of times t_i such that $\lim_{i\to\infty} |\varphi_{t_i}|_{C^0(M)} = \infty$.

Choose $S^t_{\nu,\beta} = a(t)\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}, \quad 0 \leq \beta \leq N$ as before. Since both $\left|\tilde{S}^t_{\nu,\beta}\right|_{h_0^\nu}$ and $\lambda_\beta(t)$ are uniformly bounded, we can assume

$$\lim_{i\to\infty} \lambda_\beta(t_i) = \bar{\lambda}_\beta, \quad \lim_{i\to\infty} \tilde{S}^{t_i}_{\nu,\beta} = \bar{S}_{\nu,\beta}, \quad \beta = 0, 1, \cdots, N.$$

Notice that $\bar{\lambda}_N = 1$.

Define $I(\alpha, t) = \displaystyle\int_M (\sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}_{\nu,\beta}\right|^2_{h_0^\nu})^{-\frac{\alpha}{\nu}} \omega^n$. Clearly, $I(\alpha, t_i) \leq \int_M \left|\tilde{S}_{\nu,N}\right|^{-\frac{2\alpha}{\nu}}_{h_0^\nu} \omega^n$. As $\bar{S}_{\nu,N} \in H^0(K_M^{-\nu})$ and $\alpha_{\nu,1} > \frac{n}{n+1}$, we can find a number $\alpha \in (\frac{n}{n+1}, \alpha_{\nu,1})$ such that $\int_M |\bar{S}_{\nu,N}|^{-\frac{2\alpha}{\nu}}_{h_0^\nu} \omega^n < C$. By the semi continuity of singularity exponent, (c.f. [DK], [Tian90]), we have

$$\limsup_{i\to\infty} I(\alpha, t_i) \leq \lim_{i\to\infty} \int_M |\tilde{S}_{\nu,N}|^{-\frac{2\alpha}{\nu}}_{h_0^\nu} \omega^n = \int_M |\bar{S}_{\nu,N}|^{-\frac{2\alpha}{\nu}}_{h_0^\nu} \omega^n < C.$$

Along a tamed flow, inequality $|\varphi - \sup_M \varphi - \dfrac{1}{\nu} \log \displaystyle\sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu}| < C$ holds. It follows that $\int_M e^{\alpha(\varphi_{t_i} - \sup_M \varphi_{t_i})}\omega^n < C$. Recall that $\dot{\varphi} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi + u_\omega$, we have

$$\frac{1}{V}\int_M e^{-\alpha(\varphi_{t_i} - \sup_M \varphi_{t_i})} \cdot e^{\varphi_{t_i} + u_\omega - \dot{\varphi}} \omega^n_{\varphi_{t_i}} < C.$$

Note that both $\dot{\varphi}$ and u_ω are uniformly bounded. It follows from the convexity of exponential map that $\alpha \sup_M \varphi_{t_i} + (1-\alpha)\frac{1}{V}\int_M \varphi_{t_i}\omega^n_{\varphi_{t_i}} < C$. In other words, it is

$$\sup_M \varphi_{t_i} < \frac{1-\alpha}{\alpha}\frac{1}{V}\int_M (-\varphi_{t_i})\omega^n_{\varphi_{t_i}} + C. \tag{9}$$

Combining this with inequality (5), we have

$$\sup_M \varphi_{t_i} < n\frac{1-\alpha}{\alpha}\sup_M \varphi_{t_i} + C.$$

Since $\alpha \in (\frac{n}{n+1}, 1)$, it follows that $\sup_M \varphi_{t_i}$ is uniformly bounded from above. Consequently, φ_{t_i} is uniformly bounded. This contradicts to our assumption for φ_{t_i}. $\square$

By more careful analysis, we can improve this theorem a little bit.

Proposition 2.8. *Let* $X_t \triangleq \frac{1}{\nu}\log(\sum_{\beta=0}^{N}\left|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}\right|^2_{h_0^\nu})$. *There is a constant* C *such that*

$$\left|\frac{1}{V}\int_M\{\sqrt{-1}\partial\varphi_{t_i}\wedge\bar{\partial}\varphi_{t_i}-\sqrt{-1}\partial X_{t_i}\wedge\bar{\partial}X_{t_i}\}\wedge\omega^{n-1}\right|<C. \tag{10}$$

Proof. Since every $\tilde{S}^{t_i}_{\nu,\beta}$ is a holomorphic section, direct calculation shows that

$$\triangle X_{t_i}\geq -R$$

where $\triangle$, R are the Laplacian operator and the scalar curvature under the metric ω. As $\triangle\varphi+n>0$, we can choose a constant C_0 such that $\triangle\varphi+\triangle X_{t_i}+C_0>0$. For the simplicity of notation, we omit the subindex t_i in the following argument. So we have conditions

$$|\varphi-\sup_M\varphi-X|<C, \quad \triangle X+\triangle\varphi+C_0>0.$$

Having these conditions at hand, direct computation gives us

$$\begin{aligned}
&\frac{1}{V}\int_M\{\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi-\sqrt{-1}\partial X\wedge\bar{\partial}X\}\wedge\omega^{n-1}\\
=&\frac{1}{V}\int_M(X-\varphi)(\triangle X+\triangle\varphi)\omega^n\\
=&\frac{1}{V}\int_M(X-\varphi+\sup_M\varphi+C)(\triangle X+\triangle\varphi)\omega^n\\
=&\frac{1}{V}\int_M(X-\varphi+\sup_M\varphi+C)(\triangle X+\triangle\varphi+C_0)\omega^n-\frac{C_0}{V}\int_M(X-\varphi+\sup_M\varphi+C)\omega^n\\
\geq&-\frac{C_0}{V}\int_M(X-\varphi+\sup_M\varphi+C)\omega^n\\
\geq&-2C_0C.
\end{aligned}$$

On the other hand, similar calculation shows

$$\begin{aligned}
&\frac{1}{V}\int_M\{\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi-\sqrt{-1}\partial X\wedge\bar{\partial}X\}\wedge\omega^{n-1}\\
\leq&-\frac{C_0}{V}\int_M(X-\varphi+\sup_M\varphi-C)\omega^n\\
\leq&\, 2C_0C.
\end{aligned}$$

Consequently, we have

$$\left|\frac{1}{V}\int_M\{\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi-\sqrt{-1}\partial X\wedge\bar{\partial}X\}\wedge\omega^{n-1}\right|\leq 2C_0C.$$

□

It follows from this Proposition that inequality (4) implies

$$\frac{1}{V}\int_M (-\varphi)\omega_\varphi^n \leq n \sup_M \varphi - \frac{n-1}{V}\int_M \sqrt{-1}\partial X \wedge \bar{\partial} X \wedge \omega^{n-1} + C. \tag{11}$$

Similar to the theorems in [Tian90], we can prove the following theorem.

Theorem 2.3. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν, M is a Fano manifold satisfying $\alpha_{\nu,2} > \frac{n}{n+1}$ and $\alpha_{\nu,1} > \frac{1}{2-\frac{n-1}{(n+1)\alpha_{\nu,2}}}$. Then along this flow, φ is uniformly bounded. In particular, this flow converges to a KE metric exponentially fast.*

Proof. Suppose not. We have a sequence of times t_i such that $\lim_{i\to\infty} |\varphi_{t_i}|_{C^0(M)} = \infty$.

As before, we have

$$\lim_{i\to\infty} \tilde{S}_{\nu,\beta}^{t_i} = \bar{S}_{\nu,\beta}, \quad \beta = 0, 1, \cdots, N; \quad \lim_{i\to\infty} \lambda_\beta(t_i) = \bar{\lambda}_\beta; \quad \bar{\lambda}_N = 1.$$

Claim 1. $\bar{\lambda}_{N-1} = 0$.

Otherwise, $\bar{\lambda}_{N-1} > 0$. Fix some $\alpha \in (\frac{n}{n+1}, \alpha_{2,\nu})$, we calculate

$$\begin{aligned}
I(\alpha, t_i) &= \int_M (\sum_{\beta=0}^{N} \left|\lambda_\beta(t)\tilde{S}_{\nu,\beta}^{t_i}\right|_{h_0^\nu}^2)^{-\frac{\alpha}{\nu}}\omega^n \\
&\leq \int_M (\left|\lambda_{N-1}(t_i)\tilde{S}_{\nu,N-1}^{t_i}\right|_{h_0^\nu}^2 + \left|\tilde{S}_{\nu,N}^{t_i}\right|_{h_0^\nu}^2)^{-\frac{\alpha}{\nu}}\omega^n \\
&\leq (\lambda_{N-1}(t_i))^{-2\alpha}\int_M (\left|\tilde{S}_{\nu,N-1}^{t_i}\right|_{h_0^\nu}^2 + \left|\tilde{S}_{\nu,N}^{t_i}\right|_{h_0^\nu}^2)^{-\frac{\alpha}{\nu}}\omega^n.
\end{aligned}$$

For simplicity of notation, we look h_0^ν as the default metric on the line bundle $K_M^{-\nu}$ without writing it out explicitly. Then semi continuity property implies

$$\lim_{i\to\infty}\int_M (\left|\tilde{S}_{\nu,N-1}^{t_i}\right|^2 + \left|\tilde{S}_{\nu,N}^{t_i}\right|^2)^{-\frac{\alpha}{\nu}}\omega^n = \int_M (\left|\bar{S}_{\nu,N-1}\right|^2 + \left|\bar{S}_{\nu,N}\right|^2)^{-\frac{\alpha}{\nu}}\omega^n < \infty.$$

It follows that

$$I(\alpha, t_i) < 2(\bar{\lambda}_{N-1})^{-\frac{2\alpha}{\nu}}\int_M (\left|\bar{S}_{\nu,N-1}\right|^2 + \left|\bar{S}_{\nu,N}\right|^2)^{-\frac{\alpha}{\nu}}\omega^n < C_\alpha.$$

Recall the definition of $I(\alpha, t_i)$, equation (8) implies $\int_M e^{-\alpha(\varphi(t_i) - \sup_M \varphi(t_i))}\omega^n < C$. Since $\alpha > \frac{n}{n+1}$, as we did in previous theorem, we will obtain the boundedness of $|\varphi_{t_i}|_{C^0(M)}$. This contradicts to the initial assumption of φ_{t_i}! Therefore $\bar{\lambda}_{N-1} = 0$ and we finish the proof of this Claim 1.

Claim 2. *For every small constant ϵ, there is a constant C such that*

$$(1-\epsilon)\alpha_{\nu,2}\sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2})\frac{1}{V}\int_M \varphi_{t_i}\omega_{\varphi_{t_i}}^n \leq -2(1-\epsilon)\frac{\alpha_{\nu,2}}{\nu}\log\lambda_{N-1}(t_i) + C. \tag{12}$$

Fix ϵ small, we have

$$\begin{aligned} I((1-\epsilon)\alpha_{\nu,2}, t_i) &= \int_M (\sum_{\beta=0}^{N}\left|\lambda_\beta \tilde{S}_{\nu,\beta}^{t_i}\right|^2)^{-\frac{(1-\epsilon)\alpha_{\nu,2}}{\nu}}\omega^n \\ &\leq \int_M \{\lambda_{N-1}(t_i)^2[\left|\tilde{S}_{\nu,N-1}^{t_i}\right|^2 + \left|\tilde{S}_{\nu,N}^{t_i}\right|^2]\}^{-\frac{(1-\epsilon)\alpha_{\nu,2}}{\nu}}\omega^n \\ &< C\lambda_{N-1}(t_i)^{-\frac{2(1-\epsilon)\alpha_{\nu,2}}{\nu}}. \end{aligned}$$

The tamed condition implies that

$$\int_M e^{-(1-\epsilon)\alpha_{\nu,2}(\varphi_{t_i} - \sup_M \varphi_{t_i})}\omega^n < C\lambda_{N-1}(t_i)^{-\frac{2(1-\epsilon)\alpha_{\nu,2}}{\nu}}.$$

Plugging the equation $\dot{\varphi} = \log\frac{\omega_\varphi^n}{\omega^n} + \varphi + u_\omega$ into the previous inequality implies

$$\begin{aligned} C\lambda_{N-1}(t_i)^{-\frac{2(1-\epsilon)\alpha_{\nu,2}}{\nu}} &> \int_M e^{-(1-\epsilon)\alpha_{\nu,2}(\varphi_{t_i} - \sup_M \varphi_{t_i})} \cdot e^{\varphi_{t_i} + u_\omega - \dot{\varphi}_{t_i}}\omega_{\varphi_{t_i}}^n \\ &= \int_M e^{(1-\epsilon)\alpha_{\nu,2}\sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2})\varphi_{t_i}} \cdot e^{u_\omega - \dot{\varphi}_{t_i}}\omega_{\varphi_{t_i}}^n \\ &\geq e^{-\|u_\omega\|_{C^0(M)} - \|\dot{\varphi}_{t_i}\|_{C^0(M)}}\int_M e^{(1-\epsilon)\alpha_{\nu,2}\sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2})\varphi_{t_i}}\omega_{\varphi_{t_i}}^n \\ &\geq e^{-\|u_\omega\|_{C^0(M)} - \|\dot{\varphi}_{t_i}\|_{C^0(M)}} \cdot V \cdot e^{(1-\epsilon)\alpha_{\nu,2}\sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2})\frac{1}{V}\int_M \varphi_{t_i}\omega_{\varphi_{t_i}}^n} \end{aligned}$$

Taking logarithm on both sides, we obtain inequality (12). This finishes the proof of Claim 2.

Claim 3. *For every small number $\epsilon > 0$, there is a constant C_ϵ such that*

$$\frac{1}{V}\int_M \sqrt{-1}\partial X_{t_i} \wedge \bar{\partial} X_{t_i} \wedge \omega^{n-1} \geq -\frac{(1-\epsilon)}{\nu}\log\lambda_{N-1}(t_i) - C_\epsilon. \tag{13}$$

This proof is the same as the corresponding proof in [Tian91]. So we omit it.

Plugging inequality (13) into inequality (11), together with inequality (12), we obtain

$$\begin{cases} \frac{1}{V}\int_M (-\varphi_{t_i})\omega_{\varphi_{t_i}}^n \leq n\sup_M \varphi_{t_i} + \frac{(n-1)(1-\epsilon)}{\nu}\log\lambda_{N-1}(t_i) + C, \\ (1-\epsilon)\alpha_{\nu,2}\sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2})\frac{1}{V}\int_M \varphi_{t_i}\omega_{\varphi_{t_i}}^n \leq -\frac{2(1-\epsilon)\alpha_{\nu,2}}{\nu}\log\lambda_{N-1}(t_i) + C. \end{cases}$$

Eliminating $\log\lambda_{N-1}(t_i)$, we have

$$\frac{1}{V}\int_M(-\varphi_{t_i})\omega^n_{\varphi_{t_i}} \leq \frac{(n+1)+(n-1)\epsilon}{((n+1)-(n-1)\epsilon)\alpha_{\nu,2}-(n-1)}\alpha_{\nu,2}\sup_M\varphi_{t_i}+C.$$

As inequality (9), we have $\sup_M\varphi_{t_i}\leq\frac{1-(1-\epsilon)\alpha_{\nu,1}}{(1-\epsilon)\alpha_{\nu,1}}\frac{1}{V}\int_M(-\varphi_{t_i})\omega^n_{\varphi_{t_i}}+C$. It follows that

$$\{1-\frac{(n+1)+(n-1)\epsilon}{((n+1)-(n-1)\epsilon)\alpha_{\nu,2}-(n-1)}\cdot\alpha_{\nu,2}\cdot\frac{1-(1-\epsilon)\nu\alpha_{\nu,1}}{(1-\epsilon)\nu\alpha_{\nu,1}}\}\frac{1}{V}\int_M(-\varphi_{t_i})\omega^n_{\varphi_{t_i}}\leq C$$

for every small constant ϵ and some big constant C depending on ϵ. Since we have $\alpha_{\nu,1}>\frac{1}{2-\frac{n-1}{(n+1)\alpha_{\nu,2}}}=\frac{A}{A+1}$ where $A=\frac{(n+1)\alpha_{\nu,2}}{(n+1)\alpha_{\nu,2}-(n-1)}$, so we can choose ϵ small enough such that

$$1-\frac{(n+1)+(n-1)\epsilon}{((n+1)-(n-1)\epsilon)\alpha_{\nu,2}-(n-1)}\cdot\alpha_{\nu,2}\cdot\frac{1-(1-\epsilon)\nu\alpha_{\nu,1}}{(1-\epsilon)\nu\alpha_{\nu,1}}>0.$$

This implies that $\frac{1}{V}\int_M(-\varphi_{t_i})\omega^n_{\varphi_{t_i}}$ is uniformly bounded. Therefore, $|\varphi_{t_i}|_{C^0(M)}$ is uniformly bounded. Contradiction! $\square$

Remark 2.1. *The methods applied in Theorem 2.2 and Theorem 2.3 originate from [Tian91].*

3 Plurianticanonical Line Bundles and Tamed Condition

In this section, we study the basic properties of normalized holomorphic section $S\in H^0(K_M^{-\nu})$ under the evolving metric ω_{φ_t} and h_t^ν.

3.1 Uniform Bounds for Plurianticanonical Holomorphic Sections

Let S be a normalized holomorphic section of $H^0(M,K_M^{-\nu})$, i.e., $\int_M|S|^2_{h_t^\nu}\omega^n_{\varphi_t}=1$. In this section, we will show both $\||S|_{h_t^\nu}\|_{C^0}$ and $\||\nabla S|_{h_t^\nu}\|_{C^0}$ are uniformly bounded.

Lemma 3.1. $\{(M^n,g(t)),0\leq t<\infty\}$ *is a Kähler Ricci flow solution. There is a constant A_0 depending only on this flow such that $|S|_{h_t^\nu}<A_0\nu^{\frac{n}{2}}$ whenever $S\in H^0(M,K^{-\nu})$ satisfies $\int_M|S|^2_{h_t^\nu}\omega^n_{\varphi_t}=1$.*

Proof. Fix a time t and do all the calculations under the metric g_t and h_t^ν. Recall we have uniform Soblev constant and weak Poincarè constant, so we can do analysis uniformly independent of time t.

Claim. *S satisfies the equation*

$$\triangle|S|^2=|\nabla S|^2-\nu R|S|^2. \tag{14}$$

This calculation can be done locally. Fix a point $x \in M$. Let U be a neighborhood of x with coordinate $\{z^1, \cdots, z^n\}$. Then $K_M^{-\nu}$ has a natural trivialization on the domain U and we can write $S = f(\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu$ for some holomorphic function f locally. For convenience, we denote $h = \det g_{k\bar{l}}$. Therefore, direct calculation shows

$$\begin{aligned}
\triangle|S|^2 &= g^{i\bar{j}}\{f\bar{f}h^\nu\}_{i\bar{j}} \\
&= g^{i\bar{j}}\{f_i\bar{f}h^\nu + \nu f\bar{f}h^{\nu-1}h_i\}_{\bar{j}} \\
&= g^{i\bar{j}}\{f_i\bar{f}_{\bar{j}}h^\nu + \nu\bar{f}h^{\nu-1}f_ih_{\bar{j}} + \nu fh^{\nu-1}\bar{f}_{\bar{j}}h_i + \nu(\nu-1)f\bar{f}h^{\nu-2}h_ih_{\bar{j}} + \nu f\bar{f}h^{\nu-1}h_{i\bar{j}}\}.
\end{aligned}$$

If we choose normal coordinate at the point x, then we have $h = 1$, $h_i = h_{\bar{j}} = 0$, $h_{i\bar{j}} = -R_{i\bar{j}}$. Plugging them into previous equality we have

$$\triangle|S|^2 = g^{i\bar{j}}\{f_i\bar{f}_{\bar{j}} - \nu f\bar{f}R_{i\bar{j}}\} = |\nabla S|^2 - \nu R|S|^2.$$

So equation (14) is proved.

From equation (14), we have

$$\int_M |\nabla S|^2 d\mu = \int_M \nu R|S|^2 d\mu \leq \nu\mathcal{B}$$

where $d\mu = \omega_{\varphi_t}^n$. Note that volume is fixed along Kähler Ricci flow solution, we can omit the volume term in Sobolev inequality by adjusting C_S. Therefore, Sobolev inequality implies

$$\begin{aligned}
\{\int_M |S|^{\frac{2n}{n-1}} d\mu\}^{\frac{n-1}{n}} &\leq C_S\{\int_M |S|^2 d\mu + \int_M |\nabla|S||^2 d\mu\} \\
&\leq C_S\{\int_M |S|^2 d\mu + \int_M |\nabla S|^2 d\mu\} \\
&\leq C_S\{1 + \nu\mathcal{B}\} < C\nu.
\end{aligned}$$

Here we use the property that $\bar{\nabla}S = 0$.

Note that we have the inequality $\triangle|S|^2 \geq -\nu R|S|^2$. Let $u = |S|^2$, we have

$$\triangle u \geq -\nu\mathcal{B}u, \quad \|u\|_{L^{\frac{n}{n-1}}} < C\nu^{\frac{1}{2}}$$

Multiplying this inequality by $u^{\beta-1}(\beta > 1)$ and integration by parts implies

$$\int_M |\nabla u^{\frac{\beta}{2}}|^2 d\mu \leq \frac{\beta^2}{4(\beta-1)} \cdot (\mathcal{B}\nu) \cdot \int_M u^\beta d\mu.$$

Combining this with Sobolev inequality yields

$$\{\int_M u^{\frac{n\beta}{n-1}} d\mu\}^{\frac{n-1}{n}} \leq C_S(1 + \frac{\beta^2\mathcal{B}\nu}{4(\beta-1)}) \int_M u^\beta d\mu \leq C\nu\beta \int_M u^\beta d\mu.$$

It follows that $\|u\|_{L^{\frac{n\beta}{n-1}}} \leq (C\nu)^{\frac{1}{\beta}} \beta^{\frac{1}{\beta}} \|u\|_{L^{\beta}}$. Let $\beta = (\frac{n}{n-1})^k$, we have

$$\|u\|_{L^\infty} \leq (C\nu)^{\sum_{k=1}^{\infty}(\frac{n-1}{n})^k} \cdot (\frac{n}{n-1})^{\sum_{k=1}^{\infty} k(\frac{n-1}{n})^k} \cdot \|u\|_{L^{\frac{n}{n-1}}} \leq C\nu^{\sum_{k=0}^{\infty}(\frac{n-1}{n})^k} = C\nu^n.$$

In other words, $\||S|\|_{L^\infty} \leq C\nu^{\frac{n}{2}}$. Let A_0 be the last C, we finish the proof. ☐

Corollary 3.1. *$\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution. Then we have*

$$F_\nu(x,t) \leq \frac{2\log A_0 + n \log \nu}{\nu} < B_0, \quad \forall\, x \in M,\ t \in [0, \infty),\ \nu \geq 1. \tag{15}$$

Here B_0 is a constant depending only on A_0.

Proof. According to the definition of F_ν, we only need to show

$$\sum_{\beta=0}^{N_\nu} \left|S^t_{\nu,\beta}\right|^2_{h^\nu_t}(x) \leq A_0^2 \nu^n$$

for every orthonormal holomorphic section basis $\{S^t_{\nu,\beta}\}_{\beta=0}^{N_\nu}$. However, fix x, by rotating basis, we can always find a basis such that

$$\left|S^t_{\nu,\beta}\right|^2_{h^\nu_t}(x) = 0, \quad 1 \leq \beta \leq N_\nu.$$

Therefore, by Lemma 3.1, we have

$$\sum_{\beta=0}^{N_\nu} \left|S^t_{\nu,\beta}\right|^2_{h^\nu_t}(x) = \left|S^t_{\nu,0}\right|^2_{h^\nu_t}(x) \leq A_0^2 \nu^n.$$

☐

Lemma 3.2. *$\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution. There is a constant A_1 depending only on this flow and ν such that $|\nabla S|_{h^\nu_t} < A_1$ whenever $S \in H^0(M, K^{-\nu})$ satisfying $\int_M |S|^2_{h^\nu_t} \omega^n_{\varphi_t} = 1$.*

Proof. Fix a time t and then do all the computations with respect to $g(t)$ and h^ν_t. Same as in the previous Lemma, we can do uniform analysis since the existence of uniform Sobolev and weak Poincarè constants.

Claim 1. *$|\nabla S|^2$ satisfies the equation*

$$\begin{aligned}\triangle |\nabla S|^2 = &|\nabla\nabla S|^2 + \nu^2 |Ric|^2 |S|^2 - \nu R |\nabla S|^2 \\ &- (2\nu - 1) R_{i\bar{k}} S_k \bar{S}_{\bar{i}} - \nu\{S R_i \bar{S}_{\bar{i}} + \bar{S} R_{\bar{i}} S_i\}.\end{aligned} \tag{16}$$

Suppose U to be a local coordinate around point x. Locally, we can rewrite

$$S = f(\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu,$$
$$\nabla S = \{f_i + \nu f(\log h)_i\} dz^i \otimes (\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu$$

where $h = \det g_{k\bar{l}}$. It follows that

$$|\nabla S|^2 = g^{i\bar{j}} h^\nu (f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu \bar{f}(\log h)_{\bar{j}}).$$

Choose normal coordinate at point x. So at point x, we have $g_{i\bar{j}} = \delta_{i\bar{j}}$, $h = 1$, $h_i = h_{\bar{i}} = (\log h)_i = (\log h)_{\bar{i}} = 0$, $h_{i\bar{j}} = (\log h)_{i\bar{j}} = -R_{i\bar{j}}$, $(\log h)_{ij} = (\log h)_{\bar{i}\bar{j}} = 0$. So we compute

$$\begin{aligned}
\triangle|\nabla S|^2 &= g^{k\bar{l}} \{g^{i\bar{j}} h^\nu (f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu \bar{f}(\log h)_{\bar{j}})\}_{k\bar{l}} \\
&= g^{k\bar{l}} \{-g^{i\bar{p}} g^{q\bar{j}} \frac{\partial g_{p\bar{q}}}{\partial z^k} h^\nu (f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu \bar{f}(\log h)_{\bar{j}}) \\
&\qquad + \nu g^{i\bar{j}} h^{\nu-1} h_k (f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu \bar{f}(\log h)_{\bar{j}}) \\
&\qquad + g^{i\bar{j}} h^\nu (f_{ik} + \nu f(\log h)_{ik} + \nu f_k (\log h)_i)(\bar{f}_{\bar{j}} + \nu \bar{f}(\log h)_{\bar{j}}) \\
&\qquad + g^{i\bar{j}} h^\nu (f_i + \nu f(\log h)_i) \nu \bar{f}(\log h)_{\bar{j}k}\}_{\bar{l}} \\
&= R_{j\bar{i}k\bar{k}} f_i \bar{f}_{\bar{j}} + \nu h_{k\bar{k}} f_i \bar{f}_{\bar{i}} \\
&\quad + \nu f(\log h)_{k\bar{k}i} \bar{f}_{\bar{i}} + \nu f_k (\log h)_{i\bar{k}} \bar{f}_{\bar{i}} + f_{ik} \bar{f}_{\bar{i}\bar{k}} \\
&\quad + \nu^2 f \bar{f} (\log h)_{i\bar{k}} (\log h)_{\bar{i}k} + \nu f_i \bar{f} (\log h)_{k\bar{k}\bar{i}} + \nu f_i \bar{f}_{\bar{k}} (\log h)_{k\bar{i}} \\
&= R_{j\bar{i}} f_i \bar{f}_{\bar{j}} - \nu R |\nabla f|^2 - \nu f R_i \bar{f}_{\bar{i}} - \nu R_{i\bar{k}} f_k \bar{f}_{\bar{i}} \\
&\quad + |\nabla\nabla f|^2 + \nu^2 |f|^2 |Ric|^2 - \nu \bar{f} f_i R_{\bar{i}} - \nu R_{i\bar{k}} f_k \bar{f}_{\bar{i}} \\
&= |\nabla\nabla S|^2 + \nu^2 |S|^2 |Ric|^2 - \nu R |\nabla S|^2 + (1 - 2\nu) R_{j\bar{i}} S_i \bar{S}_{\bar{j}} - \nu (S R_i \bar{S}_{\bar{i}} + \bar{S} R_{\bar{i}} S_i).
\end{aligned}$$

So we finish the proof of Claim 1.

Claim 2. *S satisfies the equation*

$$S_{,i\bar{j}} = -\nu S R_{i\bar{j}} \tag{17}$$

Suppose U to be a normal coordinate around point x. Locally, we can rewrite

$$S = f(\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu,$$
$$\nabla S = \{f_i + \nu f(\log h)_i\} dz^i \otimes (\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu,$$
$$\bar{\nabla} S = f_{\bar{i}} d\bar{z}^i \otimes (\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu = 0.$$

Recall $\Gamma^k_{ij} = g^{k\bar{l}}\frac{\partial g_{i\bar{l}}}{\partial z^j}$, it vanishes at point x. So $(\log h)_i$, $(\log h)_{ij}$ vanish at point x. Note that f is holomorphic, and our connection is compatible both with the metric and the complex structure. So $\bar{\nabla}\nabla S$ has only one term

$$\begin{aligned}\bar{\nabla}\nabla S &= \{\nu f(\log h)_{i\bar{j}}\}d\bar{z^j} \otimes dz^i \otimes (\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu \\ &= -\nu f R_{i\bar{j}} d\bar{z^j} \otimes dz^i \otimes (\frac{\partial}{\partial z^1} \wedge \cdots \frac{\partial}{\partial z^n})^\nu.\end{aligned}$$

It follows that $S_{,i\bar{j}} = -\nu S R_{i\bar{j}}$. Claim 2 is proved.

Claim 3. *There is a constant C such that $\||\nabla S|\|_{L^{\frac{2n}{n-1}}} < C$ uniformly.*

Integrate both sides of equation (16) and we have

$$\int_M |\nabla\nabla S|^2 d\mu \le \int_M \nu R|\nabla S|^2 d\mu + (2\nu - 1)\int_M R_{i\bar{k}} S_k \bar{S}_{\bar{i}} d\mu + \nu \int_M \{S R_i \bar{S}_{\bar{i}} + \bar{S} R_{\bar{i}} S_i\} d\mu$$

where $d\mu = \omega^n_{\varphi_t}$. Recall that $R_{i\bar{k}} = g_{i\bar{k}} - \dot{\varphi}_{i\bar{k}}$. It follows that

$$\begin{aligned}\int_M |\nabla\nabla S|^2 d\mu &\le \int_M \nu R|\nabla S|^2 d\mu + (2\nu - 1)\int_M |\nabla S|^2 d\mu - (2\nu - 1)\int_M \dot{\varphi}_{i\bar{k}} S_k \bar{S}_{\bar{i}} d\mu \\ &\qquad + 2\int_M \{-\nu R|\nabla S|^2 + \nu^2 R^2 |S|^2\} d\mu \\ &= 2\nu^2 \int_M R^2 |S|^2 d\mu - \nu \int_M R|\nabla S|^2 d\mu \\ &\qquad + (2\nu - 1)\int_M |\nabla S|^2 d\mu + (2\nu - 1)\int_M \dot{\varphi}_i \{S_{,k\bar{k}} \bar{S}_{\bar{i}} + S_k \bar{S}_{,\bar{i}\bar{k}}\} d\mu\end{aligned}$$

Note that we used the property $S_{,l\bar{k}} = -\nu S R_{l\bar{k}}$.

It follows that

$$\begin{aligned}\int_M |\nabla\nabla S|^2 d\mu &\le 2\nu^2 \int_M R^2|S|^2 d\mu - \nu \int_M R|\nabla S|^2 d\mu + (2\nu - 1)\int_M |\nabla S|^2 d\mu \\ &\qquad - (2\nu - 1)\nu \int_M S R \dot{\varphi}_i \bar{S}_{\bar{i}} d\mu + (2\nu - 1)\int_M \bar{S}_{,\bar{i}\bar{k}} \dot{\varphi}_i S_k d\mu \\ &\qquad \text{(Recall that } R, \dot{\varphi}, |\nabla\dot{\varphi}|, |S|, \int_M |S|^2 d\mu \text{ and } \int_M |\nabla S|^2 d\mu \text{ are all bounded.)} \\ &\le C\{1 + \int_M |\nabla S| d\mu + \int_M |\nabla\nabla S||\nabla S| d\mu\} \\ &\qquad \text{(Using Hölder inequality and interperlation: } xy \le x^2 + y^2) \\ &\le C\{1 + V + \int_M |\nabla S|^2 d\mu + \frac{1}{2C}\int_M |\nabla\nabla S|^2 d\mu + 2C\int_M |\nabla S|^2 d\mu\} \\ &= \frac{1}{2}\int_M |\nabla\nabla S|^2 d\mu + C\{1 + V + (2C + 1)\mathcal{B}\nu\}.\end{aligned}$$

By redefining C, we have proved that $\int_M |\nabla\nabla S|^2 d\mu < C$ uniformly. On the other hand, we know

$$\int_M \left|\bar{\nabla}\nabla S\right|^2 d\mu = \int_M \nu^2 |S|^2 |Ric|^2 d\mu < C \int_M |Ric|^2 d\mu < C.$$

Therefore Sobolev inequality tells us that

$$\begin{aligned}(\int_M |\nabla S|^{\frac{2n}{n-1}} d\mu)^{\frac{n-1}{n}} &\leq C_S\{\int_M |\nabla S|^2 d\mu + \int_M |\nabla|\nabla S||^2\} d\mu \\ &\leq C\{\int_M |\nabla S|^2 d\mu + \int_M |\nabla\nabla S|^2 d\mu + \int_M \left|\bar{\nabla}\nabla S\right|^2\} d\mu.\end{aligned}$$

This means $\||\nabla S|\|_{L^{\frac{2n}{n-1}}}$ is uniformly bounded along Kähler Ricci flow. So we have finished the proof of the Claim 3.

Fix $\beta > 1$, multiplying $-|\nabla S|^{2(\beta-1)}$ to both sides of equation (16) and doing integration yields

$$\begin{aligned}&\frac{4(\beta-1)}{\beta^2} \int_M \left|\nabla|\nabla S|^\beta\right|^2 d\mu \\ = &-\int_M (\nu^2 |Ric|^2 |S|^2 + |\nabla\nabla S|^2)|\nabla S|^{2(\beta-1)} d\mu + \int_M \nu R |\nabla S|^{2\beta} d\mu \\ &+ \underbrace{\int_M (2\nu-1) R_{i\bar{k}} S_k \bar{S}_{\bar{i}} |\nabla S|^{2(\beta-1)} d\mu}_{I} + \underbrace{\nu \int_M \{S R_i \bar{S}_{\bar{i}} + \bar{S} R_{\bar{i}} S_i\} |\nabla S|^{2(\beta-1)} d\mu}_{II}.\end{aligned}$$

Plugging $R_{i\bar{k}} = g_{i\bar{k}} - \dot{\varphi}_{i\bar{k}}$ into I yields

$$I = (2\nu-1) \int_M |\nabla S|^{2\beta} d\mu - (2\nu-1) \int_M \dot{\varphi}_{i\bar{k}} S_k \bar{S}_{\bar{i}} |\nabla S|^{2(\beta-1)} d\mu$$

Since $S_{,l\bar{k}} = -\nu S R_{l\bar{k}}$, using the uniformly boundedness of $\dot{\varphi}$,R and $|S|$, we have

$$\begin{aligned}\frac{I}{2\nu-1} &= \int_M |\nabla S|^{2\beta} d\mu - \int_M \dot{\varphi}_i (\nu R S \bar{S}_{\bar{i}} - S_k \bar{S}_{,\bar{i}\bar{k}}) |\nabla S|^{2(\beta-1)} d\mu \\ &\quad + (\beta-1) \int_M \dot{\varphi}_i S_k \bar{S}_{\bar{i}} |\nabla S|^{2(\beta-2)} (-\nu S R_{l\bar{k}} \bar{S}_{\bar{l}} + S_l \bar{S}_{,\bar{l}\bar{k}}) d\mu \\ &\leq \int_M |\nabla S|^{2\beta} d\mu + C\nu \int_M |\nabla S|^{2\beta-1} d\mu + C \int_M |\nabla\nabla S||\nabla S|^{2\beta-1} d\mu \\ &\quad + C(\beta-1)\nu \int_M |Ric||S||\nabla S|^{2\beta-1} d\mu + C(\beta-1) \int_M |\nabla\nabla S||\nabla S|^{2\beta-1}.\end{aligned}$$

Therefore, for some constant C (It may depends on ν), we have

$$I \leq C \int_M (|\nabla S|^{2\beta-1} + |\nabla S|^{2\beta}) d\mu + \beta C \int_M (\nu |Ric||S| + |\nabla\nabla S|) |\nabla S|^{2\beta-1} d\mu.$$

Direct calculation shows

$$
\begin{aligned}
II &= \nu \int_M \{-R|\nabla S|^{2\beta} + \nu |S|^2 R^2 |\nabla S|^{2(\beta-1)}\} d\mu \\
&\quad - \nu \int_M RS\bar{S}_{\bar{i}}(\beta-1)|\nabla S|^{2(\beta-2)}(-\nu \bar{S} R_{i\bar{l}} S_l + \bar{S}_{\bar{l}} S_{,li}) d\mu \\
&\quad + \nu \int_M \{-R|\nabla S|^{2\beta} + \nu |S|^2 R^2 |\nabla S|^{2(\beta-1)}\} du \\
&\quad - \nu \int_M R\bar{S} S_i(\beta-1)|\nabla S|^{2(\beta-2)}(-\nu S R_{l\bar{i}} \bar{S}_{\bar{l}} + S_l \bar{S}_{,\bar{l}\bar{i}}) d\mu \\
&= -2\nu \int_M R|\nabla S|^{2\beta} d\mu + 2\nu^2 \int_M |S|^2 R^2 |\nabla S|^{2(\beta-1)} d\mu \\
&\quad + 2(\beta-1)\nu^2 \int_M R|S|^2 R_{i\bar{l}} S_l \bar{S}_{\bar{i}} |\nabla S|^{2(\beta-2)} d\mu \\
&\quad - (\beta-1)\nu \int_M \{S_{,li}\bar{S}_{\bar{i}}\bar{S}_{\bar{l}} S + \bar{S}_{,\bar{l}\bar{i}} S_l S_i \bar{S}\} R|\nabla S|^{2(\beta-2)} d\mu \\
&\le C \int_M \{|\nabla S|^{2\beta} + |\nabla S|^{2(\beta-1)}\} d\mu + \beta C \int_M (\nu |Ric||S| + |\nabla\nabla S|)|\nabla S|^{2(\beta-1)} d\mu.
\end{aligned}
$$

Combining this estimate with the estimate of I we have

$$
\begin{aligned}
&\frac{4(\beta-1)}{\beta^2} \int_M \left|\nabla |\nabla S|^{\beta}\right|^2 d\mu \\
&\le - \int_M (\nu^2 |Ric|^2 |S|^2 + |\nabla\nabla S|^2)|\nabla S|^{2(\beta-1)} d\mu \\
&\quad + C \int_M \{|\nabla S|^{2\beta} + |\nabla S|^{2(\beta-1)}\} d\mu \\
&\quad + \beta C \int_M (\nu |Ric||S| + |\nabla\nabla S|)\{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta-1}\} d\mu
\end{aligned} \tag{18}
$$

Since $\beta C\nu |Ric||S||\nabla S|^{2(\beta-1)} = (\nu |Ric||S||\nabla S|^{(\beta-1)}) \cdot (\beta C |\nabla S|^{(\beta-1)})$, we see

$$
\int_M \beta C \nu |Ric||S||\nabla S|^{2(\beta-1)} d\mu \le \int_M \frac{1}{2}\nu^2 |Ric|^2 |S|^2 |\nabla S|^{2(\beta-1)} d\mu + \int_M \frac{1}{2}(\beta C)^2 |\nabla S|^{2(\beta-1)} d\mu
$$

Similar deduction yields

$$
\begin{aligned}
&\beta C \int_M (\nu |Ric||S| + |\nabla\nabla S|)\{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta-1}\} d\mu \\
&\le \int_M (\nu^2 |Ric|^2 |S|^2 + |\nabla\nabla S|^2)|\nabla S|^{2(\beta-1)} d\mu + \beta^2 C^2 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\} d\mu.
\end{aligned}
$$

By adjusting constant C, it follows from (18) that

$$
\frac{4(\beta-1)}{\beta^2} \int_M \left|\nabla |\nabla S|^{\beta}\right|^2 d\mu \le \beta^2 C^2 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\} d\mu.
$$

If $\beta \geq \frac{n}{n-1}$, we have

$$\int_M \left|\nabla|\nabla S|^{\beta}\right|^2 d\mu \leq (C\beta)^3 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\} d\mu.$$

Sobolev inequality tells us that

$$\begin{aligned}(\int_M |\nabla S|^{\beta\cdot\frac{2n}{n-1}})^{\frac{n-1}{n}} &\leq C_S\{\int_M |\nabla S|^{2\beta} d\mu + \int_M \left|\nabla|\nabla S|^{\beta}\right|^2 d\mu\} \\ &\leq (2C\beta)^3 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\} d\mu. \qquad (19)\end{aligned}$$

From this inequality and the fact $\||\nabla S|\|_{L^{\frac{2n}{n-1}}}$ is uniformly bounded, standard Moser iteration technique tells us $\||\nabla S|\|_{L^\infty} < A_1$ for some uniform constant A_1.

□

3.2 Convergence of Plurianticanonical Holomorphic Sections

In this subsection we use L^2-estimate for $\bar{\partial}$-operator to study the convergence of plurianticanonical bundles. This section is very similar to Section 5 of Tian's paper [Tian90]. For the readers' and ourselves' convenience, we write down the arguments in detail.

First let's list the important $\bar{\partial}$-lemma without proof.

Proposition 3.1 (c.f.[Tian90], Proposition 5.1.)**.** *Suppose (M^n, g, J) is a complete Kähler manifold, ω is metric form compatible with g and J, L is a line bundle on M with the hermitian metric h, and ψ is a smooth function on M. If*

$$Ric(h) + Ric(g) + \sqrt{-1}\partial\bar{\partial}\psi \geq c_0\omega$$

for some uniform positive number c_0 at every point. Then for any smooth L-valued $(0,1)$-form v on M with $\bar{\partial}v = 0$ and $\int_M |v|^2 d\mu_g$ finite, there exists a smooth L-valued function u on M such that $\bar{\partial}u = v$ and

$$\int_M |u|^2 e^{-\psi} d\mu_g \leq \frac{1}{c_0}\int_M |v|^2 e^{-\psi} d\mu_g$$

where $|\cdot|$ is the norm induced by h and g.

In our application, we fix M to be a Fano manifold, $L = K_M^{-\nu}$ for some integer ν.

This Proposition assures that the plurigenera is a continuous function in a proper moduli space of complex varieties under Cheeger-Gromov topology.

Theorem 3.1. *(M_i, g_i, J_i) is a sequence of Fano manifolds satisfying*

(a). There is an a priori constant $\mathcal{B}$ such that

$$C_S((M_i, g_i)) + \|R_{g_i}\|_{C^0(M_i)} + \|u_i\|_{C^0(M_i)} < \mathcal{B}.$$

Here $C_S((M_i, g_i))$ is the Sobolev constant of (M_i, g_i), R_{g_i} is the scalar curvature, $-u_i$ is the normalized Ricci potential. In other words, it satisfies

$$Ric_{g_i} - \omega_{g_i} = -\sqrt{-1}\partial\bar{\partial}u_i, \quad \frac{1}{V_{g_i}}\int_{M_i} e^{-u_i} d\mu_{g_i} = 1.$$

(b). There is a constant K such that $K^{-1}r^{2n} \leq \mathrm{Vol}(B(x,r)) \leq Kr^{2n}$ for every geodesic ball $B(x,r) \subset M_i$ satisfying $r \leq 1$.

(c). $(M_i, g_i, J_i) \xrightarrow{C^\infty} (\hat{M}, \hat{g}, \hat{J})$ where $(\hat{M}, \hat{g}, \hat{J})$ is a Q-Fano normal variety.

Then for any fixed positive integer ν, we have

1. *If $S_i \in H^0(M_i, K_{M_i}^{-\nu})$ and $\int_{M_i} |S_i|^2 d\mu_{g_i} = 1$, then by taking subsequence if necessary, we have $\hat{S} \in H^0(\hat{M}, K_{\hat{M}}^{-\nu})$ such that*

$$S_i \xrightarrow{C^\infty} \hat{S}, \quad \int_{\hat{M}} \left|\hat{S}\right|^2 d\mu_{\hat{g}} = 1.$$

2. *If $\hat{S} \in H^0(\hat{M}, K_{\hat{M}}^{-\nu})$ and $\int_{\hat{M}} \left|\hat{S}\right|^2 d\mu_{\hat{g}} = 1$, then there is a subsequence of holomorphic sections $S_i \in H^0(M_i, K_{M_i}^{-\nu})$ and $\int_{M_i} |S_i|^2 d\mu_{g_i} = 1$ such that $S_i \xrightarrow{C^\infty} \hat{S}$.*

Proof. For simplicity, we let $\nu = 1$. Let $\mathcal{P}$ be the singular set of $\hat{M}$. As $\hat{M}$ is normal variety, Hausdorff dimension of $\mathcal{P}$ is not greater than $2n-4$. In virtue of condition (b) and (c), volume converges as M_i converge to $\hat{M}$. Consequently, $K^{-1}r^{2n} \leq \mathrm{Vol}(B(x,r)) \leq Kr^{2n}$ holds for every geodesic ball $B(x,r) \subset \hat{M}$ satisfying $r \leq 1$. Therefore, by the fact that $\dim(\mathcal{P}) \leq 2n-4$, the Hausdorff dimension definition and packing ball method implies that there is a constant $\mathcal{V}$ such that $\mathrm{Vol}(B(\mathcal{P}, r)) \leq \mathcal{V}r^4$ whenever r is small. Now we prove part 1 and part 2 respectively.

Part1. "$\Longrightarrow$"

According to the proof of Lemma 3.1, we see there is an a priori bound A_0 such that $\||S_i|\|_{C^0(M_i)} < A_0$.

Fix any small number δ and define $U_\delta = \hat{M} \backslash B(\mathcal{P}, \delta)$. By the definition of smooth convergence, there exists a sequence of diffeomorphisms $\phi_i : U_\delta \to \phi_i(U_\delta) \subset M_i$ satisfying the following properties

(1) $\phi_i^* g_i \xrightarrow{C^\infty} \hat{g}$ uniformly on U_δ;

(2) $(\phi_i^{-1})_* \circ J_i \circ (\phi_i)_* \xrightarrow{C^\infty} \hat{J}$ uniformly on U_δ.

For convenience, define $(\phi_i)^* S_i \triangleq (\phi_i^{-1})_* S_i$. Clearly, $((\phi_i)^* S_i)|_{U_\delta}$ is a section of $(T^{(1,0)}\hat{M} \oplus T^{(0,1)}\hat{M})|_{U_\delta}$ where $T^{(1,0)}\hat{M}$ and $T^{(0,1)}\hat{M}$ are divided by the complex structure $\hat{J}$. Note that $\||((\phi_i)^* S_i)|_{U_\delta}|\|_{C^0(U_\delta)} < A_0$ and $((\phi_i)^* S_i)|_{U_\delta}$ is holomorphic under the complex structure $(\phi_i^{-1})_* \circ J_i \circ (\phi_i)_*$. By Cauchy's integration formula, all covariant derivatives of $((\phi_i^{-1})_* S_i)|_{U_\delta}$ with respect to $(\phi_i)^* g_i$ are uniformly bounded in the domain $U_{2\delta}$. Therefore there must exist a limit section $\hat{S}_{2\delta} \in (T^{(1,0)}\hat{M} \oplus T^{(0,1)}\hat{M})|_{U_{2\delta}}$ and $(\phi_i)^* S_i \xrightarrow{C^\infty} \hat{S}_{2\delta}$ on $U_{2\delta}$. This section $\hat{S}_{2\delta}$ is automatically holomorphic with respect to $\hat{J}$ since $(\phi_i^{-1})_* \circ J_i \circ (\phi_i)_* \xrightarrow{C^\infty} \hat{J}$ on $U_{2\delta} \subset U_\delta$.

As $(M_i, g_i, J_i) \xrightarrow{C^\infty} (\hat{M}, \hat{g}, \hat{J})$, we have $\lim\limits_{i\to\infty} V_{g_i}(M_i \backslash \phi_i(U_{2\delta})) < 2\mathcal{V}(2\delta)^4 = 32\mathcal{V}\delta^4$. It follows that

$$\begin{aligned}
1 \geq \int_{U_{2\delta}} |(\phi_i)^* S_i|^2 d\mu_{\phi_i^* g_i} &= \int_{\phi_i(U_{2\delta})} |S_i|^2 d\mu_{g_i} \\
&= \int_{M_i} |S_i|^2 d\mu_{g_i} - \int_{M_i \backslash \phi_i(U_{2\delta})} |S_i|^2 d\mu_{g_i} \\
&> 1 - 32A_0^2 \mathcal{V}\delta^4.
\end{aligned}$$

Therefore, for each δ, there is a limit holomorphic section $\hat{S}_{2\delta} \in H^0(\hat{U}_{2\delta}, K_{U_{2\delta}}^{-1})$ satisfying

$$\|\left|\hat{S}_{2\delta}\right|\|_{C^0(U_{2\delta})} \leq A_0, \quad 1 \geq \int_{U_{2\delta}} \left|\hat{S}_{2\delta}\right|^2 d\mu_{\hat{g}} \geq 1 - 32A_0^2 \mathcal{V}\delta^4.$$

Let $\delta = \delta_k = 2^{-k} \to 0$ and then take diagonal sequence, we obtain a subsequence of sections $(\phi_{i_k}^{-1})_* S_{i_k}|_{U_{2\delta_k}}$ satisfying

$$(\phi_{i_k}^{-1})_* S_{i_k}|_K \xrightarrow{C^\infty} \hat{S}|_K, \quad \forall \text{ compact set } K \subset \hat{M} \backslash \mathcal{P}.$$

This exactly means that $(\phi_{i_k}^{-1})_* S_{i_k} \xrightarrow{C^\infty} \hat{S}$ on $\hat{M} \backslash \mathcal{P}$. As $\hat{M}$ is a Q-Fano normal variety, $\hat{S}$ can be naturally extended to a holomorphic section of $H^0(K_{\hat{M}}^{-1})$. Moreover, we have

$$\int_{\hat{M}} \left|\hat{S}\right|^2 d\mu = \int_{\hat{M} \backslash \mathcal{P}} \left|\hat{S}\right|^2 d\mu = 1,$$

where the metric on $K_{\hat{M}}^{-\nu}$ is naturally $(\det \hat{g})^\nu$. So we finish the proof of part 1.

Part2. "$\Longleftarrow$"

Fix two small positive numbers r, δ satisfying $r \gg 2\delta$. Define function η_δ to be a cutoff function taking value 1 on $U_{2\delta}$ and 0 inside $B(\mathcal{P}, \delta)$. η_δ also satisfies $|\nabla \eta_\delta|_{\hat{g}} < \frac{2}{\delta}$.

Like before, there exists a sequence of diffeomorphisms $\phi_i : U_\delta \to \phi_i(U_\delta) \subset M_i$ satisfying the following properties

(1) $\phi_i^* g_i \xrightarrow{C^\infty} \hat{g}$ uniformly on U_δ;

(2) $(\phi_i^{-1})_* \circ J_i \circ (\phi_i)_* \xrightarrow{C^\infty} \hat{J}$ uniformly on U_δ.

$\phi_{i*}(\eta_\delta \hat{S})$ can be looked as a smooth section of the bundle $\Lambda^n(T^{(1,0)}M_i \oplus T^{(0,1)}M_i)$ by natural extension. Let π_i be the projection from $\Lambda^n(T^{(1,0)}M_i \oplus T^{(0,1)}M_i)$ to $\Lambda^n T^{(1,0)}M_i$ and denote $V_{\delta,i} = \pi_i(\phi_{i*}(\eta_\delta \hat{S}))$. The smooth convergence of complex structures implies that $V_{\delta,i}$ is an almost holomorphic section of $\Lambda^n T^{(1,0)}M_i$. In other words,

$$\lim_{i\to\infty} \sup_{\phi_i(U_{2\delta})} \left|\bar{\partial} V_{\delta,i}\right| = \lim_{i\to\infty} \sup_{\phi_i(U_{2\delta})} \left|\bar{\partial}(\pi_i(\phi_i^*(\eta_\delta \hat{S})))\right| = 0. \tag{20}$$

Here $\bar{\partial}$ is calculated under the complex structure J_i.

Notice that $V(B(\mathcal{P},\delta)) \leq \mathcal{V}\delta^4$ when δ small. Denote $\mathcal{A} = \|\left|\hat{S}\right|\|_{C^0(\hat{M})}$. Note that $\mathcal{A}$ depends on $\hat{M}$ and $\hat{S}$ itself. We have

$$1 \geq \lim_{i\to\infty} \int_{M_i} |V_{\delta,i}|^2 d\mu_{g_i} = \lim_{i\to\infty} \int_{M_i} \left|\pi_i(\phi_{i*}(\eta_\delta \hat{S}))\right|^2 d\mu_{g_i} \geq 1 - 2\mathcal{A}^2\mathcal{V}(2\delta)^4 = 1 - 32\mathcal{A}^2\mathcal{V}\delta^4.$$

Recall $V_{\delta,i}$ vanishes on $B(\mathcal{P},\delta)$, so we have

$$\int_{M_i} \left|\bar{\partial} V_{\delta,i}\right|^2 d\mu_{g_i} = \int_{\phi_i(U_{2\delta})} \left|\bar{\partial} V_{\delta,i}\right|^2 d\mu_{g_i} + \int_{\phi_i(U_\delta \backslash U_{2\delta})} \left|\bar{\partial} V_{\delta,i}\right|^2 d\mu_{g_i}.$$

By virtue of inequality (20) and the fact $|\nabla \eta_\delta|_{\hat{g}} < \frac{2}{\delta}$, $\mathrm{Vol}(\phi_i(U_\delta \backslash U_{2\delta})) \leq 2\mathcal{V}(2\delta)^4$, we obtain

$$\int_{M_i} \left|\bar{\partial} V_{\delta,i}\right|^2 d\mu_{g_i} \leq 1000\mathcal{A}^2\mathcal{V}\delta^2$$

for large i.

Let h_i be the hermitian metric on $K_{M_i}^{-1}$ induced by g_i. Clearly, we have

$$Ric(h_i) + Ric(g_i) + \sqrt{-1}\partial\bar{\partial}(-2u_i) = 2(Ric(g_i) - \sqrt{-1}\partial\bar{\partial}u_i) = 2\omega_{g_i}.$$

So we are able to apply Proposition 3.1 and obtain a smooth section $W_{\delta,i}$ of $K_{M_i}^{-1}$ such that

$$\begin{cases} \bar{\partial} W_{\delta,i} = \bar{\partial} V_{\delta,i} \\ \int_{M_i} |W_{\delta,i}|^2 e^{2u_i} d\mu_{g_i} \leq \frac{1}{2} \int_{M_i} \left|\bar{\partial} V_{\delta,i}\right|^2 e^{2u_i} d\mu_{g_i} \leq \frac{e^{2\mathcal{B}}}{2} \int_{M_i} \left|\bar{\partial} V_{\delta,i}\right|^2 d\mu_{g_i} < 500\mathcal{A}^2\mathcal{V}e^{2\mathcal{B}}\delta^2. \end{cases} \tag{21}$$

Triangle inequality implies

$$1 + \sqrt{500\mathcal{A}^2\mathcal{V}e^{2\mathcal{B}}\delta^2} > (\int_{M_i} |V_{\delta,i} - W_{\delta,i}|^2 d\mu_{g_i})^{\frac{1}{2}} > \sqrt{1 - 32\mathcal{A}^2\mathcal{V}\delta^4} - \sqrt{500\mathcal{A}^2\mathcal{V}e^{2\mathcal{B}}\delta^2}. \tag{22}$$

Therefore $S_{\delta,i} = \dfrac{V_{\delta,i} - W_{\delta,i}}{(\int_{M_i} |V_{\delta,i} - W_{\delta,i}|^2 d\mu_{g_i})^{\frac{1}{2}}}$ is a well defined holomorphic section of $K_{M_i}^{-1}$.

Direct computation shows that $W_{\delta,i}$ satisfies the elliptic equation:

$$\begin{aligned}
\triangle(|W_{\delta,i}|^2) &= |\nabla W_{\delta,i}|^2 + |\bar{\nabla} W_{\delta,i}|^2 - R|W_{\delta,i}|^2 + 2Re < W_{\delta,i}, \bar{\partial}^* \bar{\partial} W_{\delta,i} > \\
&= |\nabla W_{\delta,i}|^2 + |\bar{\nabla} W_{\delta,i}|^2 - R|W_{\delta,i}|^2 + 2Re < W_{\delta,i}, \bar{\partial}^* \bar{\partial} V_{\delta,i} > \\
&\geq |\nabla W_{\delta,i}|^2 + |\bar{\nabla} W_{\delta,i}|^2 - (R+1)|W_{\delta,i}|^2 - \left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2 \\
&\geq |\nabla W_{\delta,i}|^2 + |\bar{\nabla} W_{\delta,i}|^2 - 2\mathcal{B}\{|W_{\delta,i}|^2 + \frac{1}{2\mathcal{B}}\left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2\}. \qquad (23)
\end{aligned}$$

All geometric quantities are computed under the metric g_i and complex structure J_i. Let $f = |W_{\delta,i}|^2 + \frac{1}{2\mathcal{B}} \sup_{\varphi_i(U_{\frac{r}{2}})} \left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2$, on $\varphi_i(U_{\frac{r}{2}})$, we have

$$\triangle f \geq -2\mathcal{B} f.$$

Applying local Moser iteration in $\phi_i(U_{\frac{r}{2}})$, we obtain

$$\begin{aligned}
\|f\|_{C^0(\varphi_i(U_r))} &\leq C'(r, \mathcal{B}, \mathcal{A}) \|f\|_{L^{\frac{n}{n-1}}(\varphi_i(U_{\frac{r}{2}}))} \\
&= C'(r, \mathcal{B}, \mathcal{A}) \{ \||W_{\delta,i}|^2\|_{L^{\frac{n}{n-1}}(\varphi_i(U_{\frac{r}{2}}))} + \frac{1}{2\mathcal{B}} \sup_{\varphi_i(U_{\frac{r}{2}})} \left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2 \}.
\end{aligned}$$

Since $\sup_{\varphi_i(U_{\frac{r}{2}})} \left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2$ tends to 0 uniformly, it follows that

$$\||W_{\delta,i}|^2\|_{C^0(\varphi_i(U_r))} \leq C''(r, \mathcal{B}, \mathcal{A}) \||W_{\delta,i}|^2\|_{L^{\frac{n}{n-1}}(\varphi_i(U_{\frac{r}{2}}))}. \qquad (24)$$

On the other hand, inequality (23) can be written as

$$|\nabla W_{\delta,i}|^2 + |\bar{\nabla} W_{\delta,i}|^2 \leq \triangle(|W_{\delta,i}|^2) + 2\mathcal{B}|W_{\delta,i}|^2 + \left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2.$$

Combining this inequality with Sobolev inequality, we can apply cutoff function on $\phi_i(U_{\frac{r}{4}} \backslash U_{\frac{r}{2}})$ to obtain

$$\||W_{\delta,i}|^2\|_{L^{\frac{n}{n-1}}(\varphi_i(U_{\frac{r}{2}}))} \leq C'''(r, \mathcal{B}, \hat{M}) \{ \||W_{\delta,i}|^2\|_{L^1(\varphi_i(U_{\frac{r}{4}}))} + \sup_{\varphi_i(U_{\frac{r}{4}})} \left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2 \}.$$

Together with inequality (24), the fact $\sup_{\varphi_i(U_{\frac{r}{4}})} \left|\bar{\partial}^* \bar{\partial} V_{\delta,i}\right|^2 \to 0$ implies that

$$\begin{aligned}
\||W_{\delta,i}|^2\|_{C^0(\varphi_i(U_r))} &\leq C''''(r, \mathcal{B}, \mathcal{A}, \hat{M}) \||W_{\delta,i}|^2\|_{L^1(\varphi_i(U_{\frac{r}{4}}))} \\
&\leq C''''(r, \mathcal{B}, \mathcal{A}, \hat{M}) \||W_{\delta,i}|^2\|_{L^1(M_i)} \\
&\leq C(r, \mathcal{B}, \mathcal{A}, \mathcal{V}, \hat{M}) \delta^2.
\end{aligned}$$

The last inequality follows from estimate (21) and the fact $|u_i| < \mathcal{B}$.

Fix r, δ and let $i \to \infty$, we have $\lim_{i\to\infty} \varphi_i^*(S_{\delta,i}) = \dfrac{\hat{S} + \hat{W}_r}{\lim_{i\to\infty} (\int_{M_i} |V_{\delta,i} - W_{\delta,i}|^2 d\mu_{g_i})^{\frac{1}{2}}}$ on domain U_r. Here $\hat{W}_r$ is a holomorphic section of $H^0(U_r, K_{U_r}^{-1})$ with $\left\|\left|\hat{W}_r\right|\right\|_{C^0(U_r)} \leq C\delta$. It follows from this and inequality (22) that $\lim_{\delta\to 0}\lim_{i\to\infty} \varphi_i^*(S_{\delta,i}) = \hat{S}$ on domain U_r. Let $\delta_k = 2^{-k}$ and take diagonal sequence, we obtain $\lim_{k\to\infty} \varphi_{i_k}^*(S_{2^{-k},i_k}) = \hat{S}$ on U_r. Then let $r = 2^{-l}$ and take diagonal sequence one more time, we obtain a sequence of holomorphic sections $S_l \triangleq S_{2^{-k_l},i_{k_l}}$ such that

$$\lim_{l\to\infty} \varphi_l^*(S_l) = \hat{S}, \qquad \text{on } \hat{M}\backslash\mathcal{P}.$$

Since every S_l is a holomorphic section (w.r.t $(\phi_l^{-1})_* \circ J_l \circ (\phi_l)_*$), Cauchy integration formula implies that this convergence is actually in C^∞-topology. $\square$

3.3 Justification of Tamed Condition

In this section, we show when the Kähler Ricci flow is tamed.

Theorem 3.2. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow satisfying the following conditions.*

- *volume ratio bounded from above, i.e., there exists a constant K such that*

$$\mathrm{Vol}_{g(t)}(B_{g(t)}(x, r)) \leq Kr^{2n}$$

  for every geodesic ball $B_{g(t)}(x, r)$ satisfying $r \leq 1$.

- *weak compactness, i.e., for every sequence $t_i \to \infty$, by passing to subsequence, we have*

$$(M, g(t_i)) \xrightarrow{C^\infty} (\hat{M}, \hat{g}),$$

  where $(\hat{M}, \hat{g})$ is a Q-Fano normal variety.

Then this flow is tamed.

Proof. Suppose this result is false. For every $p_i = i!$, F_{p_i} is an unbounded function on $M \times [0, \infty)$. By Corollary 3.1, F_{p_i} has no lower bound. Therefore, there exists a point (x_i, t_i) such that

$$F_{p_i}(x_i, t_i) < -p_i. \tag{25}$$

By weak compactness, we can assume that

$$(M, g(t_i)) \xrightarrow{C^\infty} (\hat{M}, \hat{g}).$$

Moreover, as $\hat{M}$ is a Q-Fano variety, we can assume $e^{\nu F_\nu(y)} = \sum_{\alpha=0}^{N_\nu} |S_{\nu,\alpha}(y)|^2_{\hat{\omega}^\nu} > c_0$ on $\hat{M}$. Applying Theorem 3.1, we have

$$\lim_{i\to\infty} e^{\nu F_\nu(x_i,t_i)} > \frac{1}{2}c_0.$$

It follows that there are holomorphic sections $S_\nu^{(t_i)} \in H^0(K_M^{-\nu})$ satisfying

$$\int_M \left|S_\nu^{(t_i)}\right|^2_{h^\nu_{t_i}} \omega^n_{t_i} = 1, \quad \left|S_\nu^{(t_i)}\right|^2_{h^\nu_{t_i}}(x_i) = e^{\nu F_\nu(x_i,t_i)} > \frac{1}{2}c_0.$$

According to Lemma 3.1, we see there is a constant C depending only on this flow such that

$$\left|S_\nu^{(t_i)}\right|_{h^\nu_{t_i}} < C\nu^{\frac{n}{2}}.$$

So we have

$$A \triangleq \int_M \left|(S_\nu^{(t_i)})^k\right|^2_{h^{k\nu}_{t_i}} \omega^n_{t_i} < VC^{2k}\nu^{nk}.$$

Therefore, $A^{-\frac{1}{2}}(S_\nu^{(t_i)})^k$ are unit sections of $H^0(K_M^{-k\nu})$. It follows that

$$e^{k\nu F_{k\nu}(x_i,t_i)} \geq \left|A^{-\frac{1}{2}}(S_\nu^{(t_i)})^k\right|^2_{h^{k\nu}_{t_i}}(x_i) \geq V^{-1}C^{-2k}\nu^{-nk}\left|(S_\nu^{(t_i)})^k\right|^2_{h^{k\nu}_{t_i}}(x_i) \geq V^{-1}C^{-2k}\nu^{-nk}(\frac{c_0}{2})^k.$$

This implies that

$$k\nu \cdot F_{k\nu}(x_i,t_i) \geq -2k\log C - nk\log\nu + k\log(\frac{c_0}{2}) - \log V$$

for large i (depending on ν) and every k. Let $k = \frac{p_i}{\nu} = \frac{i!}{\nu}$, by virtue of inequality (25), we have

$$-k^2\nu^2 = -p_i^2 > p_i F_{p_i}(x_i,t_i) = k\nu \cdot F_{k\nu}(x_i,t_i) \geq -2k\log C - nk\log\nu + k\log(\frac{c_0}{2}).$$

However, this is impossible for large k! ☐

In Theorem 4.4 of [CW3], we have proved the weak compactness property of Kähler Ricci flow on Fano surfaces, i.e., every sequence of evolving metrics of a Kähler Ricci flow solution on a Fano surface subconverges to a Kähler Ricci soliton orbifold in Cheeger-Gromov topology. Moreover, the volume ratio upper bound is proved as a lemma to prove weak compactness. As an application of this property, we obtain

Corollary 3.2. *If $\{(M^2, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow on a Fano surface M^2, then it is a tamed Kähler Ricci flow .*

Proof. According to Theorem 4.4 of [CW3], every weak limit $\hat{M}$ is a Kähler Ricci soliton orbifold. It has positive first Chern class and it can be embedded into projective space by its plurianticanonical line bundle sections (c.f. [Baily]). In particular, every $\hat{M}$ is a Q-Fano normal variety. So Theorem 3.2 applies. $\square$

In [RZZ], Weidong Ruan, Yuguang Zhang and Zhenlei Zhang proved that the Riemannian curvature is uniformly bounded along the Kähler Ricci flow if $\int_M |Rm|^n d\mu$ is uniformly bounded. Under such condition, every sequential limit is a smooth Kähler Ricci soliton manifold, therefore Theorem 3.2 applies and we have

Corollary 3.3. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow along a Fano manifold M^n and $n \geq 3$. If*

$$\sup_{0 \leq t < \infty} \int_M |Rm|^n_{g(t)} d\mu_{g(t)} < \infty,$$

then $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a tamed Kähler Ricci flow .

4 Kähler Ricci Flow on Fano Surfaces

In this section, we give an application of the theorems we developed.

4.1 Convergence of 2-dimensional Kähler Ricci Flow

As the convergence of 2-dimensional Kähler Ricci flow was studied in [CW1] and [CW2] for all cases except $c_1^2(M) = 1$ or 3, we will concentrate on these two cases in this section.

Lemma 4.1. *Suppose M is a Fano surface, $S \in H^0(K_M^{-\nu})$, $x \in M$.*

- *If $c_1^2(M) = 1$, then $\alpha_x(S) \geq \frac{5}{6\nu}$ for every $S \in H^0(K_M^{-\nu}), x \in M$.*
- *If $c_1^2(M) = 3$, then $\alpha_x(S) \geq \frac{2}{3\nu}$ for every $S \in H^0(K_M^{-\nu}), x \in M$. Moreover, if $\alpha_x(S_1) = \alpha_x(S_2) = \frac{2}{3\nu}$, then $S_1 = \lambda S_2$ for some constant λ.*

As a direct corollary, we have

Lemma 4.2. *Suppose M is a Fano surface, ν is any positive integer.*

- *If $c_1^2(M) = 1$, then $\alpha_{\nu,1} \geq \frac{5}{6}$.*
- *If $c_1^2(M) = 3$, then $\alpha_{\nu,1} = \frac{2}{3}$, $\alpha_{\nu,2} > \frac{2}{3}$.*

Because of Lemma 4.2 and Corollary 3.2, we are able to apply Theorem 2.2 and Theorem 2.3 respectively to obtain the following theorem.

Theorem 4.1. *If M is a Fano surface with $c_1^2(M) = 1$ or $c_1^2(M) = 3$, then the Kähler Ricci flow on M converges to a KE metric exponentially fast.*

Combining this with the result in [CW1] and [CW2], we have proved the following result by Ricci flow method.

Theorem 4.2. *Every Fano surface M has a KRS metric in its canonical class. This KRS metric is a KE metric if and only if $Aut(M)$ is reductive.*

In particular, we have proved the Calabi conjecture on Fano surfaces by flow method. This conjecture was first proved by Tian in [Tian90] via continuity method.

Remark 4.1. *In [Chl], Cheltsov proved the following fact. Unless M is a cubic surface with bad symmetry and with Eckardt point (a point passed through by three exceptional lines), then there exists a finite group G such that $\alpha_G(M,\omega) > \frac{2}{3}$ for every M satisfying $c_1^2(M) \leq 5$. Using this fact, we obtain the convergence of Kähler Ricci flow on M directly if $M \sim \mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$. We thank Tian and Cheltsov for pointing this out to us. However, for the consistency of our own programme, we still give an independent proof for the convergence of Kähler Ricci flow on $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$ without applying this fact.*

4.2 Calculation of Local α-invariants

In this subsection, we give a basic proof of Lemma 4.1.

4.2.1 Local α-invariants of Anticanonical Holomorphic Sections

Proposition 4.1. *Let $S \in H^0(\mathbb{CP}^2, 3H)$, $Z(S)$ be the divisor generated by S, $x \in Z(S)$. Then $\alpha_x(S)$ is totally determined by the singularity type of x. It is classified as in the table 2.*

$\alpha_x(S)$	Singularity type of x	S's typical local equation
1	smooth	z
	transversal intersection of two lines	zw
	transversal intersection of a line and a conic curve	zw
	ordinary double point	$z^2 - w^2(w+1)$
$\frac{5}{6}$	cusp	$z^3 - w^2$
$\frac{3}{4}$	tangential intersection of a line and a conic curve	$z(z+w^2)$
$\frac{2}{3}$	intersection of three different lines	$zw(z+w)$
$\frac{1}{2}$	a point on a double line	z^2
$\frac{1}{3}$	a point on a triple line	z^3

Table 2: Local α invariants of holomorphic anticanonical sections on projective plane

Proof. Direct computation. □

Proposition 4.2. *Suppose M to be a Fano surface and $M = \mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$. $S \in H^0(M, K_M^{-1})$. Then $\alpha_x(S) \geq \frac{2}{3}$ for every $x \in Z(S)$. Moreover, if both S_1 and $S_2 \in H^0(M, K_M^{-1})$, $\alpha_x(S_1) = \alpha_x(S_2) = \frac{2}{3}$. Then there exists a nonzero constant λ such that $S_1 = \lambda S_2$.*

Proof. Let M to be $\mathbb{CP}^2$ blowup at points $p_1, \cdots, p_6$ in generic positions. Let $\pi : M \to \mathbb{CP}^2$ to be the inverse of blowup process. If $S \in H^0(M, K_M^{-1})$, then $\pi_*(Z(S))$ must be a cubic curve γ (maybe reducible) in $\mathbb{CP}^2$ and it must pass through every point p_i. It cannot contain any triple line. Otherwise, assume it contains a triple line connecting p_1 and p_2. Then $Z(S) = 3H - aE_1 - bE_2$ for some $a, b \in \mathbb{Z}^+$. On the other hand, we know $Z(S) = 3H - \sum_{i=1}^{6} E_i$. Contradiction!

Since no three p_i's are in a same line, similar argument shows that there is no double line in $\pi_*(Z(S))$.

So the table 2 implies $\alpha_x(S) = \alpha_{\pi(x)}(\pi_*(S)) \geq \frac{2}{3}$ whenever $\pi(x) \in \mathbb{CP}^2\backslash\{p_1, \cdots, p_6\}$. Therefore we only need to consider singular point $x \in \pi^{-1}(\{p_1, \cdots, p_k\})$. Without loss of generality, we assume $x \in \pi^{-1}(p_1)$ and x is a singular point of $Z(S)$. We consider this situation by the singularity type of $\pi_*(x)$. Actually, x is a singular point of $Z(S)$ only if $\pi_*(x)$ is a singular point of $\pi_*(Z(S))$. By table 2, we have the following classification.

1. $\pi_*(x) = p_1$ is an intersection point of three different lines. This case cannot happen. If such three lines exist, one of them must pass through 3 blowup points. Impossible.

2. $\pi_*(x)$ is an intersection point of two different lines. In this case, x must be a transversal intersection of a curve and the exceptional divisor E_1. Therefore, $\alpha_x(S) = 1$.

3. $\pi_*(x)$ is a cusp point. In this case, x must be a tangential intersection of a smooth curve and the exceptional divisor E_1. Moreover, the tangential order is just 1. So $\alpha_x(S) = \frac{3}{4}$.

4. $\pi_*(x)$ is a tangential intersection point of a line and a conic curve. In this case, x is the transversal intersection point of three curves $\gamma_1, \gamma_2, \gamma_3$. Moreover, they have particular properties. $[\gamma_1] = E_1, [\gamma_2] \sim 2H - \sum_{l=1}^{6} E_l + E_j, [\gamma_3] \sim H - E_1 - E_j$ for some $j \in \{2, \cdots, 6\}$. x is the intersection of 3 exceptional lines. Clearly, $\alpha_x(S) = \frac{2}{3}$.

Therefore, no matter whether $x \in \pi^{-1}\{p_1, \cdots, p_6\}$, we see $\alpha_x(S) \geq \frac{2}{3}$. Moreover, $\alpha_x(S) = \frac{2}{3}$ only if x is the transversal intersection of three exceptional lines.

It is well known that M is a cubic surface, there are totally 27 exceptional lines on M. Every point can be passed through by at most three exceptional lines. Therefore, if $\alpha_x(S_1) = \alpha_x(S_2) = \frac{2}{3}$, then $Z(S_1) = Z(S_2)$ as union of three exceptional lines passing through x. So there is a nonzero constant λ such that $S_1 = \lambda S_2$.

□

Proposition 4.3. *Suppose M to be a Fano surface and $M \sim \mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$. $S \in H^0(M, K_M^{-1})$. Then $\alpha_x(S) \geq \frac{5}{6}$ for every $x \in Z(S)$.*

Proof. Same notation as in proof of Proposition 4.2, we see $\pi_*(Z(S))$ is a cubic curve. Suppose $\pi_*(Z(S))$ is reducible, then $Z(S) = \gamma_1 + \gamma_2$ with γ_1 a line and γ_2 a conic curve. So $Z(S)$ can pass at most $2 + 5 = 7$ points of the blowup points. On the other hand, it must pass through all of them. Contradiction! Therefore, $\pi_*(Z(S))$ is irreducible.

If $\pi(x) \in \mathbb{CP}^2\backslash\{p_1, \cdots, p_8\}$, we have $\alpha_x(S) = \alpha_{\pi(x)}(\pi_*(S)) \geq \frac{5}{6} > \frac{2}{3}$ by Table 2. Suppose $\pi(x) \in \{p_1, \cdots, p_8\}$. As the 8 points are in generic position, we know no cubic curve pass through 7 of them with one point doubled. $\pi_*(Z(S))$ is a cubic curve passing all these 8 points, so it must pass through every point smoothly. As $Z(S)$ is irreducible, x must be a smooth point on $Z(S)$. So $\alpha_x(S) = 1$.

In short, $\alpha_x(S) \geq \frac{5}{6}$. $\square$

4.2.2 Local α-invariant of Pluri-anticanonical Holomorphic Sections

Proposition 4.4. *If f, g are holomorphic functions (or holomorphic sections of a line bundle) defined in a neighborhood of x, then $\alpha_x(fg) \geq \frac{\alpha_x(f)\alpha_x(g)}{\alpha_x(f)+\alpha_x(g)}$, i.e.,*

$$\frac{1}{\alpha_x(fg)} \leq \frac{1}{\alpha_x(f)} + \frac{1}{\alpha_x(g)}. \tag{26}$$

Proof. Without loss of generality, we can assume $\alpha_x(f), \alpha_x(g) < \infty$. For simplicity of notation, let $a = \alpha_x(f), b = \alpha_x(g), c = \frac{ab}{a+b}$. We only need to prove $\alpha_x(fg) \geq c$.

Fix a small number $\epsilon > 0$, note that $\frac{c}{a} + \frac{c}{b} = 1$, Hölder inequality implies

$$\int_U (fg)^{-2c(1-\epsilon)} d\mu = (\int_U f^{-2a(1-\epsilon)} d\mu)^{\frac{c}{a}} (\int_U g^{-2b(1-\epsilon)})^{\frac{c}{b}} < \infty,$$

where U is some neighborhood of x. Therefore, $\alpha_x(fg) \geq c(1-\epsilon)$. As ϵ can be arbitrarily small, we have $\alpha_x(fg) \geq c$. $\square$

As an application of Proposition A.1.1 of [Tian90], we list the following property without proof.

Proposition 4.5. *Suppose f is a holomorphic function vanishing at x with order k. In a small neighborhood, we can express f as*

$$f = a_{ij} z_1^i z_2^j + \cdots$$

Without loss of generality, we can assume that there is a pair (i, j) such that $i \geq j$, $i+j = k$ and $a_{ij} \neq 0$. Then $\alpha_x(f) \geq \frac{1}{i}$.

Lemma 4.3. *Suppose M is a cubic surface, $S \in H^0(K_M^{-m})$, $x \in M$. If $\alpha_x(S) \leq \frac{2}{3m}$, then $\alpha_x(S) = \frac{2}{3m}$, $S = (S')^m$ where $S' \in H^0(K_M^{-1})$ and $Z(S')$ is the union of three lines passing through x.*

Proof. We will prove this statement by induction. Suppose we have already proved it for all $k \le m-1$, now we show it is true for $k = m$.

Claim 1. *If S splits off an anticanonical holomorphic section S', then $Z(S')$ must be a union of three lines passing through x. Moreover, $S = (S')^m$.*

Suppose $S = S'S_{m-1}$ where $S' \in H^0(K_M^{-1})$ and $S_{m-1} \in H^0(K_M^{-(m-1)})$. Since $\alpha_x(S') \ge \frac{2}{3}$ and $\alpha_x(S_{m-1}) \ge \frac{2}{3(m-1)}$ by induction assumption, inequality (26) implies

$$\frac{3m}{2} \le \frac{1}{\alpha_x(S)} \le \frac{1}{\alpha_x(S')} + \frac{1}{\alpha_x(S_{m-1})} \le \frac{3}{2} + \frac{3(m-1)}{2} = \frac{3m}{2}.$$

It forces that

$$\alpha_x(S) = \frac{2}{3m}, \quad \alpha_x(S') = \frac{2}{3}, \quad \alpha_x(S_{m-1}) = \frac{2}{3(m-1)}.$$

Therefore the induction hypothesis tells us that $Z(S')$ is the union of three lines passing through x, $S_{m-1} = (S'')^{m-1}$ and $Z(S'')$ is the union of three lines passing through x. As there are at most three lines passing through x on a cubic surface, we see $Z(S') = Z(S'')$. By changing coefficients if necessary, we have $S_{m-1} = (S')^{m-1}$. It follows that $S = (S')^m$ and we have finished the proof.

Claim 2. *There must be a line passing through x.*

Otherwise, there is a pencil of anticanonical divisors passing through x. In this pencil, a generic divisor is irreducible and it vanishes at x with order 2. Choose such a divisor and denote it as $Z(S')$. Locally, we can represent S by a holomorphic function f, as $\alpha_x(f) = \alpha_x(S) \le \frac{2}{3m}$, we see $mult_x(f) \ge \lceil \frac{3m}{2} \rceil$. If m is odd, then $Z(S') \nsubseteq Z(S)$ will imply

$$3m = K_M^{-1} \cdot K_M^{-1} \ge 2mult_x(f) \ge 3m+1.$$

Impossible! Since $Z(S')$ is irreducible, we know $Z(S') \subset Z(S)$. Therefore, $S = S'S_{m-1}$. According to Claim 1, $Z(S')$ is union of three lines and therefore $Z(S')$ is reducible. This contradicts to the assumption of $Z(S')$.

So m must be an even number and $mult_x(f) = \frac{3m}{2}$ exactly. Now f can be written as

$$\sum_{i,j \ge 0} a_{ij} z_1^i z_2^j, \quad a_{ij} = 0 \quad \text{whenever} \quad i+j < \frac{3m}{2}.$$

Using the fact $\alpha_x(f) \le \frac{2}{3m}$, Proposition 4.5 implies $a_{i,j} = 0$ whenever $i < \frac{3m}{2}$, $a_{\frac{3m}{2},0} \ne 0$. Therefore locally f can be written as $a_{\frac{3m}{2},0} z_1^{\frac{3m}{2}} \cdot h$ for some nonzero holomorphic function h. This means that $Z(S)$ contains a curve with multiplicity $\frac{3m}{2}$. This is impossible for $S \in K_M^{-m}$ as M is the blown up of six generic points on $\mathbb{CP}^2$.

Claim 3. *The number of lines passing through x is greater than 1.*

Otherwise, there is exactly one line L_1 passing through x. So there is an irreducible degree 2 curve D passing through x such that $L_1 + D = Z(S')$ for some $S' \in H^0(K_M^{-1})$. Locally, we can write S as $l_1 h$ where l_1 is the defining function for L_1. As $\alpha_x(l_1) = 1$, Hölder inequality implies that $\alpha_x(h) \le \frac{2}{3m-2}$. Consequently, $mult_x(h) \ge \lceil \frac{3m}{2} \rceil - 1$. If $2L_1 \not\subseteq Z(S)$, we have

$$m + 1 = (K_M^{-m} - L_1) \cdot L_1 \ge \{h = 0\} \cdot L_1 \ge \lceil \frac{3m}{2} \rceil - 1 \quad \Leftrightarrow m \le 4.$$

If $m > 4$, this inequality is wrong so we have $2L_1 \subset Z(S)$. Actually, using this argument and induction, we can show that $\lceil \frac{m}{4} \rceil L_1 \subset Z(S)$.

For simplicity of notation, let $p = \lceil \frac{m}{4} \rceil$. Locally, S can be written as $l_1^p h$. Clearly, $\alpha_x(h) \le \frac{2}{3m-2p}$ and $mult_x(h) \ge \lceil \frac{3m}{2} \rceil - p$. Let f_q and h_{q-2} be the lowest degree term of f and h respectively. Then we may assume that $h_{q-2} = z_1^{j_1} z_2^{j_2} + \cdots$ and any term $z_1^i z_2^j$ in h_{q-2} satisfying $i \ge j_1$. Now we have two cases to consider.

Case1. L_1 is tangent to $\{z_1 = 0\}$.

If $(p+1)L_1 \not\subseteq Z(S)$, then

$$m + p = (K_M^{-m} - pL_1) \cdot L_1 \ge \{h = 0\} \cdot L_1 \ge (\lceil \frac{3m}{2} \rceil - p) \cdot 2, \quad \Leftrightarrow p \ge \frac{m + 2\lceil \frac{m}{2} \rceil}{3}.$$

Here we use the fact $j_1 \ge \lceil \frac{3m}{2} \rceil - p$ since $\alpha_x(h) \le \frac{2}{3m-2p}$. This contradicts to our definition $p = \lceil \frac{m}{4} \rceil$. Therefore, $(p+1)L_1 \subset Z(S)$.

Case2. L_1 is not tangent to $\{z_1 = 0\}$.

In this case, $f_q = \lambda z_1^{a_1} z_2^{a_2+p} + \cdots$ for some $\lambda \ne 0$. Moreover, every $z_1^i z_2^j$ in f_q satisfies $i \ge j_1$. Therefore, the fact $\alpha_x(S) \le \frac{2}{3m}$ and Proposition 4.5 implies $a_1 \ge \lceil \frac{3m}{2} \rceil$. It follows that $q \ge p + \lceil \frac{3m}{2} \rceil$. Under these conditions, if $(p+1)L_1 \not\subseteq Z(S)$, we have

$$m + p = (K_M^{-m} - pL_1) \cdot L_1 \ge \lceil \frac{3m}{2} \rceil \Leftrightarrow p \ge \lceil \frac{m}{2} \rceil.$$

Impossible if $m \ge 3$. So $(p+1)L_1 \subset Z(S)$. If $m \le 2$, as $\lceil \frac{m}{2} \rceil = \lceil \frac{m}{4} \rceil = 1$, we already know $\lceil \frac{m}{2} \rceil L_1 \subset Z(S)$.

Therefore by repeatedly rewriting S in local charts and considering case 1 and case 2, we can actually prove that and $\lceil \frac{m}{2} \rceil L_1 \subset Z(S)$. For simplicity, let $n = \lceil \frac{m}{2} \rceil$. Moreover, we have following conditions:

Suppose S can be written as $l_1^n h'$ locally. Then either $(n+1)L_1 \subset Z(S)$ or L_1 is not tangent to $\{z_1 = 0\}$.

From here, we can show $D \subset Z(S)$. In fact, if $(n+1)L_1 \subset Z(S)$ and $D \not\subseteq Z(S)$, we have

$$2m = K_M^{-m} \cdot D \ge (n+1)L_1 \cdot D + mult_x(h'') \ge 2(n+1) + (\lceil \frac{3m}{2} \rceil - (n+1)) \Leftrightarrow m \ge 2n + 1,$$

where h'' is the function such that locally S is represented by $l_1^{n+1} h''$. This inequality is impossible as $n = \lceil \frac{m}{2} \rceil$. If L_1 is not tangent to $\{z_1 = 0\}$, we know $mult_x(h') \ge n + \lceil \frac{3m}{2} \rceil =$

$m+2n$. Therefore, $D \nsubseteq Z(S)$ implies that

$$2m = K_M^{-m} \cdot D \geq nL_1 \cdot D + mult_x(h') \geq m + 4n \Leftrightarrow m \geq 4\lceil \frac{m}{2} \rceil.$$

Impossible! Therefore, no matter which case happens, we have $D \subset Z(S)$. So $D + L_1 \subset Z(S)$. It follows that S splits off an $S' \in H^0(K_M^{-1})$ with $Z(S') = L_1 + D$, this contradicts to Claim 1!

Claim 4. *The number of lines passing through x is greater than* 2.

Otherwise, there are only two lines L_1 and L_2 passing through x. There is a unique line L_3 not passing through x such that $L_1 + L_2 + L_3 \in K_M^{-1}$. We first prove the following property:

$$k(L_1 + L_2) \subset Z(S) \quad \text{for all} \quad 0 \leq k \leq n = \lceil \frac{m}{2} \rceil.$$

Actually, by induction, we can assume $(k-1)(L_1+L_2) \in Z(S)$. Then S can be represented by a holomorphic function $f = l_1^{k-1} l_2^{k-1} h$ locally. Note that $\alpha_x(l_1^{k-1} l_2^{k-1}) = \frac{1}{k-1}$, Hölder inequality implies $\alpha_x(h) \leq \frac{\frac{2}{3m}}{1-\frac{2(k-1)}{3m}} = \frac{2}{3m-2(k-1)}$. It follows that

$$mult_x(h) \geq \lceil \frac{3m}{2} \rceil + 1 - k = m + n + 1 - k.$$

If $kL_1 \nsubseteq Z(S)$, we have

$$m = (K_M^{-m} - (k-1)(L_1 + L_2)) \cdot L_1 \geq \{h = 0\} \cdot l_1 = 0 \geq mult_x(h) \geq m + n + 1 - k \Leftrightarrow k \geq n + 1.$$

This contradicts to the assumption of k. Therefore, we have $kL_1 \subset Z(S)$. Similarly, $kL_2 \subset Z(S)$. So $k(L_1 + L_2) \subset Z(S)$.

Now locally S can be written as $l_1^n l_2^n h$. We have $\alpha_x(h) \leq \frac{2}{3m-2n}$, $mult_x(h) \geq \lceil \frac{3m}{2} - n \rceil = m$. Assume $mult_x(h) = m$. Under a local coordinates, $h = \sum_{i,j \geq 0} a_{ij} z_1^i z_2^j$. According to the fact $\alpha_x(h) \leq \frac{2}{3m-2n}$, Proposition 4.5 implies that $a_{ij} = 0$ whenever $i < \lceil \frac{3m}{2} - n \rceil = m$. Since $mult_x(h) = m$, we see that the lowest homogeneous term of f is of form $l_1^n l_2^n z_1^m$. The condition $\alpha_x(S) \leq \frac{2}{3m} < \frac{1}{m}$ implies that either L_1 or L_2 is tangent to $\{z_1 = 0\}$ at x. Suppose L_1 does so. If $(n+1)L_1 \nsubseteq Z(S)$, we have

$$\begin{aligned} m + n = (K_M^{-m} - nL_1) \cdot L_1 &\geq \{l_2^n h = 0\} \cdot L_1 \\ &\geq n + \{\sum_{i,j \geq 0} a_{ij} z_1^i z_2^j = 0\} \cdot L_1 \geq n + \inf\{2i + j | a_{ij} \neq 0\} \geq n + 2m. \end{aligned}$$

Impossible! It follows that $(n+1)L_1 + nL_2 \subset Z(S)$.

Consider L_3. If $L_3 \nsubseteq Z(S)$, we have

$$m = L_3 \cdot K_M^{-m} \geq ((n+1)L_1 + nL_2) = 2n + 1.$$

This absurd inequality implies $L_3 \subset Z(S)$. Let $S' \in K_M^{-1}$ such that $Z(S') = L_1 + L_2 + L_3$. So have split S as $S = S' S_{m-1}$. However, $Z(S')$ is not the union of three lines passing

through x. This contradicts to Claim 1!

So there must exist three lines $L_1, L_2, L_3 \subset Z(S)$ passing through x. Since M is a cubic surface, there exists an $S' \in H^0(K_M^{-1})$ such that $Z(S') = L_1 + L_2 + L_3$. As we argued in Claim 4, $L_1, L_2, L_3 \subset Z(S)$. Therefore $L_1 + L_2 + L_3 \subset Z(S)$ and S splits off an anticanonical holomorphic section S'. By Claim 1, we have $S = (S')^m$. $\square$

Similarly, we can prove the following property by induction.

Lemma 4.4. *Suppose M is a Fano surface and $M \sim \mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$, $S \in H^0(K_M^{-m})$, $x \in M$. Then $\alpha_x(S) \geq \frac{5}{6m}$ for every $x \in M$.*

Proof. Suppose we have proved this statement for all $k \leq m-1$.

Suppose this statement doesn't hold for $k = m$, then there is a holomorphic section $S \in H^0(K_M^{-m})$ and point $x \in M$ such that $\alpha_x(S) < \frac{5}{6m}$. Let f be a local holomorphic function representing S. Clearly, $mult_x(f) > \frac{6m}{5}$. Choose $S' \in H^0(K_M^{-1})$ such that $x \in Z(S')$. Since S' is irreducible, if $Z(S') \not\subset Z(S)$, we have

$$m = Z(S) \cdot Z(S') > \frac{6m}{5}.$$

It is impossible! Therefore, $Z(S') \subset Z(S)$. It follows that $S = S' S_{m-1}$ for some $S_{m-1} \in H^0(K_M^{-1})$. So Proposition 4.5 implies

$$\alpha_x(S) \geq \frac{\frac{5}{6} \cdot \frac{5}{6(m-1)}}{\frac{5}{6} + \frac{5}{6(m-1)}} = \frac{5}{6m}.$$

This contradicts to the assumption of $\alpha_x(S)$! $\square$

Lemma 4.1 is the combination of Lemma 4.3 and Lemma 4.4.

References

[An89] Michael T. Anderson, Ricci Curvature Bounds and Einstein Metrics on Compact Manifolds, Journal of the American Mathematical Society, Vol.2, No.3.(Jul.,1989), 455-490.

[An90] Michael T. Anderson, Convergence and rigidity of manifolds under Ricci curvature bounds, Inventiones mathematicae, Vol.102(1990), 429-445.

[Baily] Walter L. Baily, On the Imbedding of V-manifolds in Projective Space, American Journal of Mathematics, 79, 403-430(1957).

[Bando90] Shigetoshi Bando, Bubbling out of Einstein manifolds, Tohoku Math. J.(2) 42(1990), no.2, 205-216.

[BKN] Shigetoshi Bando, Atsushi Kasue, Hiraku Nakajima, On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth, Invent. Math. 97(1989), no.2, 313-349.

[Ca] Eugenio Calabi, Proceedings of the International Congress of Mathematicians (Amsterdam, 1954), Vol. 2, pp.206-207, Noordhoff, Groningen, 1954.

[Cao85] Huaidong Cao, Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. Invent. Math.81(1985), no.2, 359-372.

[Ch] Jeff Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math. 92(1970), 61-74.

[CH] Xiuxiong Chen, Haozhao Li, Stability of Kähler Ricci flow, arXiv:0801.3086.

[ChT] Jeff Cheeger, Gang Tian, Curvature and Injectivity Radius Estimates for Einstein 4-manifolds, J. Amer. Math. Soc., Volume 19, Number 2, 487-525.

[Chl] Ivan Cheltsov, Log Canonical Threshholds of Del Pezzo surfaces, arXiv:math/0703175.

[ChS] Ivan Cheltsov, Constantin Shramov, Log canonical thresholds of smooth Fano threefolds. With an appendix by Jean-Pierre Demailly, arXiv:0809.3963.

[CLW] Xiuxiong Chen, Claude LeBrun, Brian Weber, On Conformally Kähler, Einstein Manifolds, arXiv:math/0609694.

[CST] Xiuxiong Chen, Song Sun, Gang Tian, A note on Kähler-Ricci soliton, arXiv:0806.2848.

[CT1] Xiuxiong Chen, Gang Tian, Ricci flow on Kähler-Einstein surfaces, Invent Math.147(2002), no.3, 487-544.

[CT2] Xiuxiong Chen, Gang Tian, Ricci flow on Kähler-Einstein manifolds, Duke Math. J.131 (2006), no.1, 17–73.

[CW1] Xiuxiong Chen, Bing Wang, Kähler Ricci Flow on Fano Surfaces(I), arXiv:0710.5204.

[CW2] Xiuxiong Chen, Bing Wang, Remarks on Kähler Ricci flow, arXiv:0809.3963.

[CW3] Xiuxiong Chen, Bing Wang, Space of Ricci flows (I), arXiv:0902.1545, submitted.

[Di] Weiyue Ding, Remarks on the existence problem of positive Kähler-Einstein metrics, Math. Ann, 282(1982), 463-471.

[Don] Simon K.Donaldson, Scalar curvature and projective embeddings, I, Journal of Differential Geometry, 59(2001), 479-522.

[DK] Jean-Pierre Demailly, János Kollár, Semi-continuity of complex singularity exponents and Kähler Einstein metrics on Fano orbifolds, Ann. Sci. École Norm. Sup.(4), 34(4):525-556, 2001.

[DT] Weiyue Ding, Gang Tian, Kähler-Einstein metrics and the generalized Futaki invariants, Invent. Math., 110(1992), 315-335.

[Fu] Akito Futaki, An obstruction to the existence of Einstein Kähler metrics. Invent. Math.73(1983), no.3, 437-443.

[FZ] Fuquan Fang, Yuguang Zhang, Convergence of Kähler Ricci flow with integral curvature bound, arXiv:0707.3470.

[Gr] Michael Gromov, Curvature, diameter and Betti numbers, Comment. Math. Helv.56(1981), no.2, 179-195.

[Ha82] Richard S.Hamilton, Three-manifolds with positive Ricci curvature, J.Differential Geometry. 17(1982), no.2, 255-306.

[Hei] Gordon Heier, Convergence of the Kähler Ricci flow and multiplier ideal sheaves on Del Pezzo surfaces, arXiv:0710.5725.

[Kosta] Dimitra Kosta, Del Pezzo surfaces with Du Val Singularities, arXiv: 0904.0943v1.

[Lu] Zhiqin Lu, On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, American Journal of Mathematics 122(2000), 235-273.

[Ma] Yozô Matsushima, Sur la structure du group d'homeomorphismes analtiques d'une certaine varietie Kaehlerinne. Nagoya Math. J.11, 145-150(1957).

[Pe] Grisha Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.

[PSS] Duong H. Phong, Natasa Sesum and Jacob Sturm, Mutiplier ideal sheaves and the Kähler Ricci flow , arXiv:math.DG/0611794.

[PSSW1] Duong H. Phong, Jian Song, Jacob Sturm and Ben Weinkove, The Kähler Ricci flow with positive bisectional curvature, arXiv: math.DG/0706.2852.

[PSSW2] Duong H. Phong, Jian Song, Jacob Sturm and Ben Weinkove, The Kähler Ricci flow and the $\bar{\partial}$ operator on vector fields, arXiv:math.DG/0705.4048.

[Ru] Yanir A. Rubinstein, On the construction of Nadel multiplier ideal sheaves and the limiting behavior of the Ricci flow, Trans. Amer. Math. Soc, vol. 361, number 11, November 2009, Pages 5839-5850.

[RZZ] Weidong Ruan, Yuguang Zhang, Zhenlei Zhang, Bounding sectional curvature along a Kähler-Ricci flow, arXiv:0710.3919.

[Se1] Natasa Sesum, Convergence of a Kähler-Ricci flow, arXiv:math/0402238.

[Se2] Natasa Sesum, Compactness results for the Kähler-Ricci flow, arXiv:0707.2974.

[SeT] Natasa Sesum, Gang Tian, Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman), J.Inst. Math. Jussieu 7(2008), no. 3, 575-587.

[Siu] Yumtong Siu, The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group, Ann. of Math. 127(1988), 585-627.

[SYl] Yalong Shi, α-invariants of smooth cubic surfaces with Eckardt points, arXiv: math.AG/0902.3203.

[Tian87] Gang Tian, On Kähler-Einstein metrics on complex surfaces with $c_1 > 0$, Invent. Math. 89(1987), no. 2, 225-246.

[Tian90] Gang Tian, On Calabi's conjecture for complex surfaces with positive first Chern class, Invent. Math. 101, no.1, 101-172.

[Tian90t] Gang Tian, On a set of polarized Kähler metrics on algebraic manifolds, Journal of Differential Geometry, 32(1990), 99-130.

[Tian91] Gang Tian, On one of Calabi's Problems, Proceedings of Symposia in Pure Mathematics, Volume 52(1991), Part2.

[Tb] Gang Tian, Canonical Metrics in Kähler Geometry, Notes taken by Meike Akveld, Lectures in Mathematics, ETH Zürich, Birkhäser Verlag, Basel, 2000.

[TV1] Gang Tian, Jeff Viaclovsky, Moduli spaces of critical Riemannian metrics in dimension four, Advances in Mathematics 196(2005), no.2, 346-372.

[TV2] Gang Tian, Jeff Viaclovsky, Volume growth, curvature decay, and critical metrics, arXiv:math/0612491.

[TY] Gang Tian, Shingtung Yau, Kähler-Einstein metrics on complex surfaces with $c_1 > 0$, Comm. Math. Phys. 112(1987), no.1, 175-203.

[TZ] Gang Tian, Xiaohua Zhu, Convergence of Kähler Ricci flow, J. Amer. Math. Soc. 20(2007).

[TZs] Gang Tian, Xiaohua Zhu, Perelman's W-functional and stability of Kähler Ricci flow , arXiv:0801.3504.

[Wang] Bing Wang, Ricci flow on Orbifold, preprint.

[Yau78] Shingtung Yau, On the Ricci curvatre of a compact Kähler manifold and the complex Monge-Ampère equation. I, Comm.Pure Appl.Math. 31(1978), no.3, 339-411.

[Ye] Rugang Ye, Entropy Functionals, Sobolev Inequalites And κ-Noncollapsing Estimates Along The Ricci Flow, arXiv:0709.2724.

[Zhq] Qi S. Zhang, A uniform Sobolev inequality under Ricci flow, arXiv:0706.1594v4.

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